Journal of Computer and Mathematical Sciences, *Vol. 6(1), 33-41, January 2015* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

The Maximal Monopoly of Graphs

Ahmed Mohammed Naji and N. D. Soner

Department of Studies in Mathematics,
University of Mysore, Manasagangotri, Mysore, INDIA.

(Received on: January 23, 2015)

ABSTRACT

In a graph $G=(V,E)$, a subset $D \subseteq V(G)$ is said to be a monopoly set of G if any vertex $v \in V - D$ has at least $\frac{d(v)}{2}$ neighbors in D, where $d(v)$ is dgree of v in G. A monopoly set D of G is called a maximal monopoly set of G if $V - D$ is not a monopoly set in G, by the maximal monopoly size of G, we mean the smallest cardinality of a maximal monopoly set in G, denoted by $Mo(G)$. In this paper, we introduce and study the concept of maximal monopoly size in graphs. We investigate the relationship between maximal monopolies size and some parameters of graphs. Bounds on $Mo(G)$ and exact values for some standard graphs are found.

Mathematics Subject Classification : 05C69, 05C38, 05C99.

Keywords: Monopoly, Maximal Monopoly, Domination, Maximal Domination, Independent.

1. INTRODUCTION

In this paper, we are conceder with a simple graph $G=(V,E)$, that nonempty, finite, have no loops no multiple and directed edges. Let G be such a graph and let n and m be the number of its vertices and edges, respectively. The degree of a vertex v in a graph G, denoted by $d(v)$, is the number of vertices adjacent to v. Denote to the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$. For any vertex v of a graph G, the open neighborhood of v is the set $N(v)=\{u \in V : uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$ the open neighborhood is the set $N(S) = \cup_{v \in S} N(v)$, the degree of a vertex $v \in V(G)$ with respect to a subset S is $d_S(v) = | N(v) \cap S |$. For a graph G, a set of vertices $S \subseteq V(G)$ is called an independent set of G if no two vertices in S are adjacent. The Independent number, denoted by $\beta(G)$, is the maximum cardinality of an independent set in G. For a graph G, a set of vertices $S \subseteq V(G)$ is called a vertex cover set of G if every edge $e \in E(G)$ has at least one end in S. The vertex cover number, denoted by $\alpha(G)$, is the minimum cardinality of vertex cover set in G. We refer the reader to[4] for graph theoretical not defined in this paper. A set D of vertices in a graph G is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G [5]. The concept of maximal domination number was introduced by V. R. Kulli *et al.*[7]. A dominating set D of G is maximal if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set. The concepts of maximal equitable domination number of a graph was introduced by P. N. Vinaykumar and D. Soner Nandappa[11] and maximal X-domination number of a bipartite graph was introduced by Y. B. Venkatakrishnan and *et al.*[10].

A subset D of vertices set of G is called a monopoly set if for every vertex $v \in V(G) - D$ has at least $\frac{d(v)}{2}$ neighbors in D, where $d(v)$ denoted to a degree of a vertex v in G, the monopoly size of G is the smallest cardinality of a monopoly set in G, denoted by $mo(G)$. In particular, monopolies are dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg[9]. For more details in dynamos in graphs (see e.g.[1,2,3,8,12]). In[6], the author defined a monopoly set of a graph G, proved that the $mo(G)$ for general graph is at least $\frac{n}{2}$, discussed the relationship between matchings and monopolies and he showed that any graph G admits a monopoly with at most $\alpha'(G)$ vertices.

In this paper, we introduces and study the concept of maximal monopoly size $Mo(G)$ of graph. We investigate the relationship between maximal monopolies size and some parameters of graph as monopolies size and maximal domination number in graphs. Bounds on $Mo(G)$ and exact values for some standard graphs are found.

Definition 1.1 A monopoly set D of a graph G is called a maximal monopoly set if

$V - D$ is not monopoly set in G. The maximal monopoly size $Mo(G)$ of G is the minimum cardinality of a maximal monopoly set.

Definition 1.2 A maximal monopoly set D of a graph G is minimal if for every vertex $v \in D$, $D - \{v\}$ is not maximal monopoly set in G.

2. EXAMPLE

For a graph G_1 as shown in Figure. 1, $D_1 = \{v_2, v_4, v_6\}$ is a minimal monopoly set and $V - D_1$ is also a monopoly set. So, D is not a maximal monopoly set in G_1. while, $D_2 = \{v_2, v_4, v_5, v_6\}$ is a monopoly set and $V - D_2$ is not a monopoly set in G_1. So, D_2 is a maximal monopoly set in G_1 and $mo(G_1) < Mo(G_1)$.

For a graph G_2 as shown in Figure. 1, $D = \{v_3, v_4, v_5, v_6\}$ is a minimal monopoly set and $V - D$ is not a monopoly set in G_2. So, D is a maximal monopoly set in G_2 and $mo(G_2) = Mo(G_2)$.

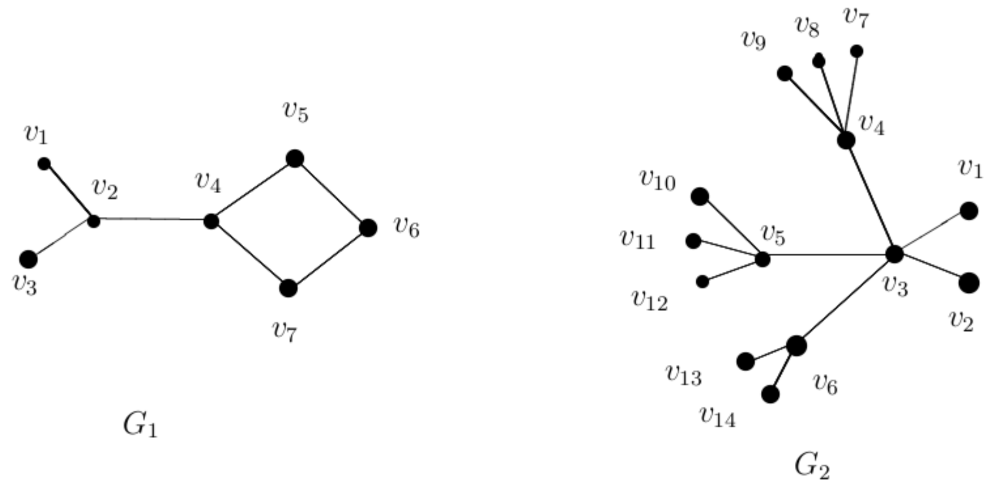

Figure 1

3. EXACT VALUES OF MAXIMAL MONOPOLY SIZE OF SOME STANDARD GRAPHS

1. For the complete graph K_n, $n \geq 2$, $Mo(K_n) = \lceil \frac{n}{2} \rceil + 1$;

2. For the path P_n, $Mo(P_n) = \lceil \frac{n}{3} \rceil + 1$
3. For the cycle C_n,

$$Mo(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 2, & if \quad n \equiv 0 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil + 1, & otherwise \ . \end{cases}$$

4. For a complete bipartite graph $K_{s,r}$ of order $s+r$,

$$Mo(K_{s,r}) = s+1, \quad s \leq r$$

5. For a wheel graph W_n of order $n \geq 4$,

$$Mo(W_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 2, & if \quad n \equiv 0 \ (mod\ 3); \\ \lfloor \frac{n}{3} \rfloor + 2, & otherwise \ . \end{cases}$$

4. PROPERTIES OF MAXIMAL MONOPOLY OF GRAPHS

In this section, we investigate relationships between maximal monopoly size and monopoly size, dominating number and maximal dominating number in graphs. We introduce some properties of a maximal monopole set of graphs. We obtain some bounds on maximal monopoly size of graphs.

Observation 4.1 For any graph G, $\gamma(G) \leq mo(G) \leq Mo(G)$.

Proof. We know that every monopoly set is a dominating set of graph and every maximal monopoly set is a monopoly set of graph. Hence, the result is holds.

Remark 4.2 The maximal monopoly size $Mo(G)$ and the maximal domination number $\gamma_m(G)$ of graphs are not comparable. For this situation consider the graphs in Figure. 2:

For the graph in Figure. 2.a, the set $\{v_1, v_2, v_3\}$ is both maximal monopoly set and maximal dominating set with minimum cardinality. Hence, $Mo(G) = \gamma_m(G) = 3$.

For the graph in Figure. 2.b, the set $\{v_1, v_2, v_3\}$ is a maximal monopoly set and the set $\{v_1, v_2\}$ is a maximal dominating set with minimum cardinality. Hence, $Mo(G) = 3$ and $\gamma_m(G) = 2$. Therefore, $\gamma_m(G) < Mo(G)$.

For the graph in Figure. 2.c, the set $\{v_1, v_2, v_3\}$ is a maximal monopoly set and the set $\{v_1, v_2, v_3, v_4\}$ is a maximal dominating set with minimum cardinality. Hence, $Mo(G) = 3$ and $\gamma_m(G) = 4$. Therefore, $Mo(G) < \gamma_m(G)$

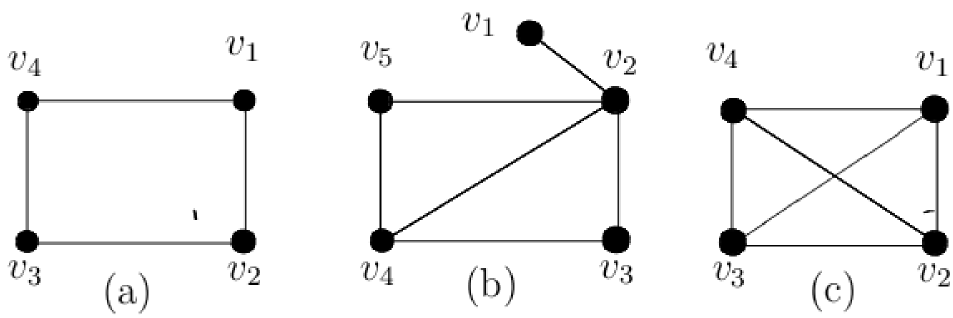

Figure 2

Theorem 4.3 Let D be a monopoly set of a graph G. Then D is a maximal monopoly set in G if and only if there exists at least a vertex $v \in D$ such that; $d_D(v) > \frac{d(v)}{2}$.

Proof. Let D be a maximal monopoly set of a graph G. Then V - D is not monopoly set. Hence, there exists a vertex $v \in D$ such that

$$| N(v) \cap (V - D) | = d_{V-D}(v) < \frac{d(v)}{2}.$$

Since $d(v) = d_D(v) + d_{V-D}(v) < d_D(v) + \frac{d(v)}{2}$,it follows that

$$\frac{d(v)}{2} < d_D(v).$$

Conversely, let D be a monopoly set of G and there exists a vertex $v \in D$ such that $d_D(v) > \frac{d(v)}{2}$. Since $d_{V-D}(v) = d(v) - d_D(v) < d(v) - \frac{d(v)}{2} = \frac{d(v)}{2}$ for some $v \in D$, it folows that $V - D$ is not monopoly set in G. Therefore, D is a maximal monopoly set in G.

Proposition 4.4 If a graph G contains an isolated vertex, then a minimal monopoly set of G is a maximal monopoly set.

Proof. Let v be an isolated vertex of G. Then v is in every monopoly set D of G. Therefore, $V - D$ is not a monopoly set. Hence, D is a maximal monopoly set of G.

Corollary 4.5 If G contains an isolated vertex, then $Mo(G) = mo(G)$.

Theorem 4.6 For any connected graph G, a maximal monopoly set of a graph G is not an independent set in G.

Proof. let D be a maximal monopoly set in G. Then from Theorem 4.2 there exists a vertex $v \in D$ such that $d_D(v) > \frac{d(v)}{2}$. On contrary, suppose that D is an independent set in G. Then, for every vertex $v \in D$, $d_D(v) = 0 < \frac{d(v)}{2}$, which is a contradiction. Then D is not independent set in G.

Theorem 4.7 For any graph G, every maximal monopoly set of G intersect with every monopoly set in G.

Proof. Let D be a maximal monopoly set of G. On contrary, suppose that there exist a monopoly set H in G such that $D \cap H = \phi$. Then $H \subseteq V - D$. Hence, $V - D$ contains a monopoly set H. Therefore, $V - D$ is itself a monopoly set in G, which is a contradiction.

Theorem 4.8 For any graph G, $Mo(G) = 1$ if and only if $G \cong K_1$.

Proof. It is clear, if $G \cong K_1$, then $Mo(G) = 1$. On the other hand, let $G \neq K_1$ (G don't equifalant K_1). Then G has at least two vertices. We consider the following two cases:

Case 1: G is a disconnected graph, then from the definition $Mo(G) \geq n \geq 2$.

Case 2: G is a connected graph, from Theorem 4.5 a maximal monopoly set D in a connected graph G is not independent set in G. Then there exist at least two adjacent vertices in D. Hence, $Mo(G) \geq 2$. Hence, if $G \neq K_1$, then $Mo(G) \neq 1$. Therefore, the result is true.

Theorem 4.9 For any nontrivial graph G, $2 \leq Mo(G) \leq n$. These bounds are sharp, the upper bound is attained if $G \cong K_2$ or K_3 and the star $K_{1,n}$ achieves the lower bound.

Theorem 4.10 For any connected graph G of order n and size m,

$$\frac{n}{m} \leq Mo(G) \leq m+1$$

Proof. Since for any connected graph $n \leq 2m$. From the previous Theorem $Mo(G) \geq 2$, Hence, $Mo(G) \geq \frac{n}{m}$. Since every connected graph with n vertices has at least $n-1$ edges. Then from the previous Theorem again $Mo(G) \leq n \leq m+1$. The graph K_2 achieves the lower and upper bounds.

Corollary 4.11 For any connected graph, $Mo(G) \leq 2m - n + 2$

Proof. By Theorem 4.8, $Mo(G) \leq n \leq 2(n-1) - n + 2 \leq 2m - n + 2$.
Therefor, the result is holds. K_2 achieves this bound.

Theorem 4.12 Let G be a connected graph. Then $Mo(G) \leq \alpha(G) + 1$

Proof. Let $S \subseteq V(G)$ be a maximal independent set in G. Then $V - S$ is a monopoly set. Thus $(V - S) \cup \{v\}$ is a maximal monopoly set in G, for every vertex $v \in S$. Hence, $Mo(G) \leq n - \beta(G) + 1$. Since for any graph G, $\beta(G) + \alpha(G) = n$. it follows that $Mo(G) \leq \alpha(G) + 1$.

Theorem 4.13 Let G be a connected graph with minimum degree $\delta(G) > 0$. Then

$$Mo(G) \leq mo(G) + \delta(G)$$

Proof. Let D be a minimum monopoly set of a connected graph G and let v be a vertex with minimum degree in G. we consider the following cases:

Case 1: $v \in D$. Then $D \cup N(v)$ is maximal monopoly set in G. Hence,

$$Mo(G) \leq |D \cup N(v)| = mo(G) + \delta(G).$$

Case 2: $v \notin D$. Then we consider the following two subcases:

Subcase 2.1: $d(v) = 1$. Then $D \cup \{v\}$ is a maximal monopoly set. Hence,

$$Mo(G) \leq |D \cup \{v\}| \leq mo(G) + 1 = mo(G) + \delta(G)$$

Subcase 2.2: $d(v) \geq 2$. Then $D \cup \{v,u\}$ (where $u \in N(v) \cap (V - D)$, if any) is a maximal monopoly set. Hence,

$$Mo(G) \leq |D \cup \{v,u\}| = mo(G) + 2 \leq mo(G) + \delta(G).$$

Theorem 4.14 For any connected graph G with minimum degree $\delta(G)$,

$$Mo(G) \geq \lceil \frac{\delta(G)}{2} \rceil + 1$$

This bound is sharp.

Proof. Let D be a maximal monopoly set of G. By Theorem 4.2, there exist a vertex $v \in D$ such that $d_D(v) > \frac{d(v)}{2}$. Hence,

$$\begin{aligned} |D| &\geq |N[v] \cap D| = d_D(v) + 1 \\ &\geq \frac{d(v)}{2} + 1 \\ &\geq \lceil \frac{\delta(G)}{2} \rceil + 1. \end{aligned}$$

Therefore, $Mo(G) \geq \lceil \frac{\delta(G)}{2} \rceil + 1$.

The complete graph K_n with even order achieves this bound.

Corollary 4.15 for any connected graph G,

$$Mo(G) \geq \lceil \frac{\kappa(G)}{2} \rceil + 1,$$

where $\kappa(G)$ is the connectivity of a graph G .

Proof. For any connected graph G , $\kappa(G) \le \delta(G)$. Then by Theorem 4.14, the result is true. The bound is attained if $G \cong P_3$ or $G \cong K_n$, n is even .

REFERENCES

1. E. Berger, Dynamic monopolies of constant size, *Journal of Combinatorial Theory, Series B*, 83, 191-200 (2001).
2. J. Bermond, J. Bond, D. Peleg and S. Perennes, The power of small coalitions in graphs, *Discrete Applied Mathematics*, 127, 399 – 414 (2003).
3. P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka and N. Santoro, On time versus size for monotone dynamic monopolies in regular topologies. *Journal of Discrete Algorithms*, 1, 129 – 150 (2003).
4. F. Harary, Graph Theory, Addison Wesley, Massachusetts, (1969).
5. T. W. Haynes, S. T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York (1998).
6. K. Khoshkhak, M. Nemati, H. Soltani, and M. Zaker, A study of monopoly in graphs, *Graph and Combinatorial Mathematics*, 29,1417 – 1427 (2013).
7. V.R. Kulli, Theory of domination in graphs, Vishwa International Publications, Gulbarga, India (2010).
8. A. Mishra and S. B. Rao, Minimum monopoly in regular and tree graphs, Discrete Mathematics, 306(14),1586 – 1594 (2006).
9. D. Peleg, Local majorities; coalitions and monopolies in graphs; a review, *Theoretical Computer Science*, 282, 231 – 257 (2002).
10. Y. B. Venkatakrishnan, C. Natarajan and S. K. Ayyaswamy, Maximal X-domination number of a bipartite graph, *European Journal of Scientific Research*, 78(4), 555-558 (2012).
11. P. N. Vinaykumar and D. N. Soner, The maximal equitable domination of a graph, *Journal of Computer and Mathematical Sciences*, 2(4), 617-620 (2011).
12. M. Zaker, On dynamic monopolies of graphs with general thresholds, *Discrete Mathematics*, 312, 1136 – 1143 (2012).