Some Properties of Fuzzy Probabilistic Automata

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ABSTRACT

This paper introduces a new type of probabilistic automaton, defined as fuzzy probabilistic automaton and its fuzzy regular language. A fuzzy probabilistic automaton is a fuzzy automaton in which, from a state, the sum of the fuzzy memberships of the transitions on an input symbol is one. Some of the closure properties on the languages generated by the fuzzy probabilistic automaton are investigated.

Keywords: fuzzy probabilistic automata, recognizability, fuzzy regular sets.

1. INTRODUCTION

Formal languages and automata theory is an integral part of theoretical Computer science. Research in this area was initiated by Noam Chomsky in earlier 1950’s, in his study on grammars and grammatical structure of a language proposed a mathematical model of a grammar. Finite automata plays a crucial role in the theory of programming languages, compiler constructions, switching circuit designing, computer controllers, neuron net, text editor and lexical analyzer1.

In1959, Rabin and Scott introduced new extended version of deterministic finite automata and presented solution to many basic decision problems. First they added the non-determinism. Then in 1963, Rabin presented in the concept of probabilistic automata2. Now, probabilistic automata is used a lot in natural language processing, string processing, compilers, modelling and verifying the non-deterministic systems3.

The probabilistic automata2 is a generalization of finite deterministic automata, where the automata have two-valued output which, when in a state with an input, the automata has a probability of going to other states.
In this paper, we introduce a concept of fuzzy probabilistic automata, where the automata have fuzzy transitions in which, from a state, the sum of the fuzzy memberships of the transitions on an input symbol is one. We discuss some of the closure properties on the languages generated by the fuzzy probabilistic automaton.

2. FUZZY PROBABILISTIC AUTOMATON

This section familiarizes the definition of fuzzy probabilistic automaton, recognizability and its fuzzy regular languages with appropriate examples.

Definition 2.1.
A fuzzy probabilistic automaton (fpa) is a 5-tuple \( M = (Q, \Sigma, \mu, i, f) \), where
\( Q \) is a finite non-empty set of states,
\( \Sigma \) is a finite non-empty set of input symbols,
\( \mu: Q \times \Sigma \times Q \rightarrow [0,1] \), called the fuzzy transition function,
\( i: Q \rightarrow [0,1] \), the fuzzy subset of initial states, defined by \( i(q) = 1 \) or \( 0 \), for each \( q \in Q \),
\( f: Q \rightarrow [0,1] \), the fuzzy subset of final states, defined by \( f(q) = 1 \) or \( 0 \), for each \( q \in Q \).
And For all \( p \in Q \) and \( a \in \Sigma \), \( \sum_{q\in Q} \mu(p, a, q) = 1 \).

Remark:
The set \( I = \{ q \in Q \mid i(q) = 1 \} \) is called fuzzy subset of initial states and the set \( F = \{ q \in Q \mid f(q) = 1 \} \) fuzzy subset of final states.

Definition 2.2.
Let \( M = (Q, \Sigma, \mu, i, f) \) be an fpa. The extended fuzzy transition function for \( M \) is the fuzzy subset \( \mu^*: Q \times \Sigma^* \times Q \rightarrow [0,1] \) has been defined as follows:
For all \( p, q \in Q, a \in \Sigma \) and \( x \in \Sigma^* \),
\( \mu^*(p, \lambda, q) = \begin{cases} 1, & \text{if } \mu(p, a, q) = 0 \\ 0, & \text{otherwise} \end{cases} \)
\( \mu^*(p, xa, q) = \bigvee \{ \mu^*(p, x, r) \land \mu^*(r, a, q) \mid r \in Q \} \)

Definition 2.3.
Let \( M = (Q, \Sigma, \mu, i, f) \) be an fpa. Let \( x \in \Sigma^* \). Then \( x \) is said to be recognized by \( M \), if \( deg_M(x) = \bigvee \{ i(p) \land \mu^*(p, x, q) \land f(q) \mid q \in Q \} > 0 \)

Definition 2.4.
Let \( M = (Q, \Sigma, \mu, i, f) \) be an fpa. Let \( L(M) = \{ x \in \Sigma^* \mid deg_M(x) > 0 \} \). Then \( L(M) \) is called the language recognized by the fpa \( M \).

Definition 2.5.
Let \( M = (Q, \Sigma, \mu, i, f) \) be an fpa. A fuzzy subset \( A \subset \Sigma^* \) is said to be recognized, if \( L(M) = A \).

Definition 2.6.
The fuzzy probabilistic language accepted by an fpa \( M = (Q, \Sigma, \mu, i, f) \) is a fuzzy subset of \( \Sigma^* \)
and is denoted by \( L_M: \Sigma^* \rightarrow [0,1] \) and defined by
\( L_M(x) = \bigvee \{ i(p) \land \mu^*(p, x, q) \land f(q) \mid q \in Q \} \mid p \in Q \)
Example 2.7.

Let $M = (Q, \Sigma, \mu, i, f)$ be an fpa, where $Q = \{q_1, q_2, q_3, q_4, q_5\}, \Sigma = \{a, b\}$ and $\mu: Q \times \Sigma \times Q \rightarrow [0, 1]$ defined as follows:

\[
\begin{align*}
\mu(q_1, a, q_1) &= 0.5 & \mu(q_3, b, q_3) &= 0.6 \\
\mu(q_1, a, q_2) &= 0.2 & \mu(q_1, b, q_4) &= 0.4 \\
\mu(q_1, a, q_4) &= 0.3 & \mu(q_2, b, q_1) &= 1 \\
\mu(q_2, a, q_3) &= 1 & \mu(q_3, b, q_3) &= 1 \\
\mu(q_3, a, q_3) &= 1 & \mu(q_4, b, q_5) &= 1 \\
\mu(q_4, a, q_1) &= 1 & \mu(q_5, b, q_5) &= 1 \\
\mu(q_5, a, q_5) &= 1 & \end{align*}
\]

$i: Q \rightarrow [0, 1]$ is defined by

\[
i(q_1) = 1 \quad i(q_2) = 1 \quad i(q_3) = 0 \quad i(q_4) = 1 \quad i(q_5) = 0
\]

$f: Q \rightarrow [0, 1]$ is defined by

\[
\begin{align*}
f(q_1) &= 0 & f(q_2) &= 0 & f(q_3) &= 1 & f(q_4) &= 0 & f(q_5) &= 1 \\
\mu(q_1, a, q_1) + \mu(q_1, a, q_2) + \mu(q_1, a, q_4) &= 0.5 + 0.2 + 0.3 = 1, \\
\mu(q_1, b, q_1) + \mu(q_3, b, q_4) &= 0.6 + 0.4 = 1, \\
\mu(q_2, a, q_3) &= 1, & \mu(q_2, b, q_1) &= 1 \\
\mu(q_3, a, q_3) &= 1, & \mu(q_3, b, q_3) &= 1 \\
\mu(q_4, a, q_1) &= 1, & \mu(q_4, b, q_5) &= 1 \\
\mu(q_5, a, q_5) &= 1, & \mu(q_5, b, q_5) &= 1
\end{align*}
\]
The fuzzy probabilistic language accepted by \( fpa \ M \) is given by

\[
L_M(x) = \begin{cases} 
0.2, & \text{if } x \in (a+b)^*aa(a+b)^* \\
1, & \text{if } x \in a(a+b)^* \\
0.4, & \text{if } x \in (a+b)^*bb(a+b)^* \\
0.3, & \text{if } x \in (a+b)^*ab(a+b)^* \\
1, & \text{if } x \in b(a+b)^* \\
0, & \text{otherwise}
\end{cases}
\]

3. CLOSURE PROPERTIES OF FUZZY REGULAR LANGUAGES

Some of the closure properties such as union, concatenation, Kleene’s closure of fuzzy regular languages on \( fpa \) are examined with examples.

**Definition 3.1.**

Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2) \) be two \( fpa \)'s such that \( Q_1 \cap Q_2 = \phi \). The union of \( M_1 \) and \( M_2 \) is the \( fpa \ M_1 \cup M_2 = (Q, \Sigma, \mu, i, f) \), where \( Q = Q_1 \cup Q_2 \), \( \mu : Q \times X \times X \rightarrow [0,1] \) is defined as follows:

For all \( p, q \in Q \)

\[
\mu(p, a, q) = \begin{cases} 
\mu_1(p, a, q), & \text{if } p, q \in Q_1 \\
\mu_2(p, a, q), & \text{if } p, q \in Q_2 \\
0, & \text{otherwise}
\end{cases}
\]

Let \( I = I_1 \cup I_2 \) be the fuzzy subset of initial states. Then \( i : Q \rightarrow [0,1] \) is defined by

\[
i(q) = \begin{cases} 
1, & \text{if } q \in I \\
0, & \text{if } q \notin I
\end{cases}
\]

Let \( F = F_1 \cup F_2 \) be the fuzzy subset of final states. Then \( f : Q \rightarrow [0,1] \) is defined by

\[
f(q) = \begin{cases} 
1, & \text{if } q \in F \\
0, & \text{if } q \notin F
\end{cases}
\]

For all \( p \in Q \) and \( a \in \Sigma \),

\[
\sum_{q \in Q} \mu(p, a, q) = \begin{cases} 
1, & \text{if } p \in Q_1 \text{ and } q \in Q_1 \\
1, & \text{if } p \in Q_2 \text{ and } q \in Q_2 \\
0, & \text{otherwise}
\end{cases}
\]

**Lemma 3.2.**

Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2) \) be two \( fpa \)'s and \( M = M_1 \cup M_2 \). Then for all \( x \in \Sigma^* \),

\[
\mu^*(p, x, q) = \begin{cases} 
\mu_1^*(p, x, q), & \text{if } p, q \in Q_1 \\
\mu_2^*(p, x, q), & \text{if } p, q \in Q_2 \\
0, & \text{otherwise}
\end{cases}
\]

**Proof.**

We prove the result by induction on the length of a string \( x \in \Sigma^*, (i.e.) \) on \( |x| = n \).
Basis for induction.
For \( n = 0 \). Then \( x = \lambda \).
\[
\mu^*(p, \lambda, q) = \begin{cases} 
1, & \text{if } \mu(p, a, q) = 0 \\
0, & \text{otherwise}
\end{cases}
\]
Hence basis for induction.
We have to notice that \((p, \lambda, q) = \begin{cases} 
1, & \text{if } \mu(p, a, q) = 0 \\
0, & \text{otherwise}
\end{cases}\), but \( \mu^*(p, \lambda, q) \) and \( \mu(p, \lambda, q) \) are not same in nature.

Assumption.
Suppose that the result is true for all \( x \in \Sigma^* \), \(|x| < n\).

Induction step.
Let \(|x| = n\).
Then \( x = ya \) for some \( y \in \Sigma^*, a \in \Sigma, |y| = n - 1 \).

Case (i).
Let \( p, q \in Q_1 \).
\[
\mu^*(p, x, q) = \mu^*(p, ya, q) \\
= V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q] \\
= (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_1]) \lor (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_2]) \lor 0 \\
= \mu_1(p, ya, q) \\
= \mu_1(p, x, q).
\]

Case (ii).
Let \( p, q \in Q_2 \).
\[
\mu^*(p, x, q) = \mu^*(p, ya, q) \\
= V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q] \\
= (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_1]) \lor (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_2]) \lor 0 \\
= \mu_2(p, ya, q) \\
= \mu_2(p, x, q).
\]

Case (iii).
Let \( p \in Q_1 \) and \( q \in Q_2 \).
\[
\mu^*(p, x, q) = \mu^*(p, ya, q) \\
= V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q] \\
= (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_1]) \lor (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_2]) \\
\text{Since } \mu(r, a, q) = 0, \text{ for all } r \in Q_1, q \in Q_2 \text{ and } \mu^*(p, y, r) = 0, \text{ for all } p \in Q_1, r \in Q_2, \text{ Therefore, } \mu^*(p, x, q) = 0.
\]

Case (iv).
Let \( p \in Q_2 \) and \( q \in Q_1 \).
\[
\mu^*(p, x, q) = \mu^*(p, ya, q) \\
= V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q] \\
= (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_1]) \lor (V[\mu^*(p, y, r) \land \mu(r, a, q)/r \in Q_2]) \\
\text{Since } \mu(r, a, q) = 0, \text{ for all } r \in Q_2, q \in Q_1 \text{ and } \mu^*(p, y, r) = 0, \text{ for all } p \in Q_2, r \in Q_1.
\]
Therefore, \( \mu^*(p, x, q) = 0 \).
Hence the lemma.

**Theorem 3.3.**
Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2) \) be two fpa’s with fuzzy languages \( L_1 \) and \( L_2 \) respectively. Then \( L \) is a fuzzy regular language accepted by \( M = M_1 \cup M_2 \), where \( L = L_1 \cup L_2 \).

**Proof.**
Let \( x \in \Sigma^* \).
\[
L(x) = \bigvee \{ \{i(p) \land \mu^*(p, x, q) \land f(q) \mid q \in Q \} / p \in Q \} \\
= \bigvee \{ \{i(p) \land \mu^*(p, x, q) \land f(q) \mid q \in Q_1 \} / p \in Q_1 \} \\
\lor \bigvee \{ \{i(p) \land \mu_1(p, x, q) \land f(q) \mid q \in Q_2 \} / p \in Q_2 \} \\
= L_1(x) \lor L_2(x) \\
= L_1 \cup L_2 
\]
Hence the theorem.

**Definition 3.4.**
Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2) \) be two fpa’s such that \( Q_1 \cap Q_2 = \emptyset \). Then the fuzzy automaton \( M_1 \cap M_2 = (Q_1 \times Q_2, \Sigma, \mu_1 \land \mu_2, i_1 \land i_2, f_1 \land f_2) \), where
The fuzzy transition function \((\mu_1 \land \mu_2)(Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2) \rightarrow [0, 1] \) is defined by
\[
(\mu_1 \land \mu_2)((p_1, p_2), a, (q_1, q_2)) = \mu_1(p_1, a, q_1) \land \mu_2(p_2, a, q_2) 
\]
The fuzzy subset of initial states \((i_1 \land i_2)(Q_1 \times Q_2) \rightarrow [0, 1] \) is defined by
\[
(i_1 \land i_2)(p_1, p_2) = i_1(p_1) \land i_2(p_2) 
\]
The fuzzy subset of final states \((f_1 \land f_2)(Q_1 \times Q_2) \rightarrow [0, 1] \) is defined by
\[
(f_1 \land f_2)(p_1, p_2) = f_1(p_1) \land f_2(p_2) 
\]

**Theorem 3.5.**
Let \( A \) and \( B \) be recognizable sets over \( \Sigma^* \) with fuzzy regular languages \( L_1 \) and \( L_2 \) which are accepted by fpa’s \( M_1 \) and \( M_2 \) respectively. Then the set \( AB \) is recognizable with fuzzy regular language \( L \) accepted by an fpa \( M \) such that \( L = L_1L_2 \).

**Proof.**
Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) and \( M_2 = (Q_2, \Sigma, \mu_2, i_2, f_2) \) be two fpa’s with their fuzzy regular languages \( L_1 \) and \( L_2 \) such that \( Q_1 \cap Q_2 = \emptyset \).
\( \text{(i.e.)} \) \( L_1: \Sigma^* \rightarrow [0, 1] \) such that \( L_1(x) > 0 \) for every \( x \in A \) and \( L_2: \Sigma^* \rightarrow [0, 1] \) such that \( L_2(x) > 0 \) for every \( x \in B \).
Let \( Q = Q_1 \cup Q_2 \).
Define a fuzzy automaton \( M = (Q, \Sigma, \mu, i, f) \), where \( \mu: Q \times \Sigma \times Q \rightarrow [0, 1] \) is defined as follows:

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For every $p, q \in Q_1$, $\mu(p, a, q) = \mu_1(p, a, q)$
For every $p, q \in Q_2$, $\mu(p, a, q) = \mu_2(p, a, q)$
For every $p \in Q_1, q \in Q_2$ and if $i_1(p) > 0$ and $i_2(q) > 0$, $\mu(p, \lambda, q) = f_1(p) \land i_2(q)$.

$i: Q \to [0, 1]$ is defined by $i(p) = \begin{cases} i_1(p), & \text{if } p \in Q_1 \\ 0, & \text{otherwise} \end{cases}$

$f: Q \to [0, 1]$ is defined by $f(p) = \begin{cases} f_2(p), & \text{if } p \in Q_2 \\ 0, & \text{otherwise} \end{cases}$

Let $F_1$ be the set of final states of $fpa M_1$ and $I_2$ the set of initial states of initial states of $fpa M_2$.

Since $f_1(p) = \begin{cases} 1, & \text{if } p \in F_1 \\ 0, & \text{if } p \notin F_1 \end{cases}$ and $i_2(q) = \begin{cases} 1, & \text{if } q \in I_2 \\ 0, & \text{if } q \notin I_2 \end{cases}$, $\mu(p, \lambda, q) = 1$.

Hence $\Sigma \mu(p, \lambda, q) = 1$ and therefore, $M$ is the $fpa$.

Let $L_3$ be the fuzzy regular language accepted by the $fpa M$ and $w \in \Sigma^*$.

For $L_3(w) = 0$, the result is obvious.

Let $L_3(w) > 0$.

Then $L_3(w) = V \{i(p) \land \mu^*(p, w, q) \land f(q)/q \in Q\}/p \in Q$.

Since $Q$ is finite and $w$ is of finite length, there exists $x = a_1 a_2 ... a_n \in A$ and

$y = b_1 b_2 ... b_m \in B$ such that $w = xy$.

Let $p_0, p_1, ..., p_n, q_0, q_1, ..., q_m \in Q$.

$L_3(w) = i(p_0) \land \mu(p_0, a_1, p_1) \land ... \land \mu(p_{n-1}, a_n, p_n)$

$\land \mu(p_n, \lambda, q_0) \land \mu(q_0, b_1, q_1) \land ... \land \mu(q_{m-1}, b_m, q_m) \land f(q_m)$

Since each term is non-negative, $\mu(p_n, \lambda, q_0) = f_1(p_n) \land i_2(q_0)$.

Therefore,

$L_3(w) = i(p_0) \land \mu(p_0, a_1, p_1) \land ... \land \mu(p_{n-1}, a_n, p_n)$

$\land f_1(p_n) \land i_2(q_0) \land \mu(q_0, b_1, q_1) \land ... \land \mu(q_{m-1}, b_m, q_m) \land f(q_m)$

$\leq \{V \{i_1(p_0) \land \mu_1(p_0, a_1, p_1) \land ... \land \mu_1(p_{n-1}, a_n, p_n) \land f_1(p_n)/p_0, p_1, ..., p_n \in Q\}\}$

$\land \{V \{i_2(q_0) \land \mu_2(q_0, b_1, q_1) \land ... \land \mu_2(q_{m-1}, b_m, q_m) \land f_2(q_m)/q_0, q_1, ..., q_m \in Q\}\}$

$L_3(w) \leq L_1(x) \land L_2(y)$.

Hence $L_3(w) \leq \{L_1(x) \land L_2(y)/w = xy, x \in A, y \in B\}$.

Let $x \in A$ and $y \in B$. Then $L_1(x) > 0$ and $L_2(y) > 0$.

$L_1(x) = V \{i_1(p_0) \land \mu_1(p_0, a_1, p_1) \land ... \land \mu_1(p_{n-1}, a_n, p_n) /p_0, p_1, ..., p_n \in Q_1\}$

Since $Q_1$ is finite, there exists $p_0, p_1, ..., p_n \in Q_1$ such that

$L_1(x) = i_1(p_0) \land \mu_1(p_0, a_1, p_1) \land ... \land \mu_1(p_{n-1}, a_n, p_n) \land f_1(p_n)$, where

$i_1(p_0) > 0, f_1(p_n) > 0$ and $x = a_1 a_2 ... a_n$

Similarly,

$L_2(y) = V \{i_2(q_0) \land \mu_2(q_0, b_1, q_1) \land ... \land \mu_2(q_{m-1}, b_m, q_m) /q_0, q_1, ..., q_m \in Q_2\}$

Since $Q_2$ is finite, there exists $q_0, q_1, ..., q_m \in Q_2$ such that

$L_2(y) = i_2(q_0) \land \mu_2(q_0, b_1, q_1) \land ... \land \mu_2(q_{m-1}, b_m, q_m) \land f_2(q_m)$, where
\[ i_2(q_0) > 0, f_2(q_m) > 0 \] and \( y = b_1 b_2 \ldots b_m \).

Therefore,

\[ L_1(x) \land L_2(y) = (i_1(p_0) \land \mu_1(p_0, a_1, p_1) \land \ldots \land \mu_1(p_{n-1}, a_n, p_n) \land f_1(p_n)) \]
\[ \land (i_2(q_0) \land \mu_2(q_0, b_1, q_1) \land \ldots \land \mu_2(q_{m-1}, b_m, q_m) \land f_2(q_m)) \]

Since \( i_2(q_0) > 0, f_2(q_m) > 0 \), \( \mu(p_n, \lambda, q_0) = f_1(p_n) \land i_2(q_0) \).

\[ L_1(x) \land L_2(y) = i(p_0) \land \mu(p_0, a_1, p_1) \land \ldots \land \mu(p_{n-1}, a_n, p_n) \]
\[ \land \mu(p_n, \lambda, q_0) \land \mu(q_0, b_1, q_1) \land \ldots \land \mu(q_{m-1}, b_m, q_m) \land f(q_m) \]
\[ = L_3(a_1 a_2 \ldots a_n \lambda b_1 b_2 \ldots b_m) \]

Hence \( \forall \{ L_1(x) \land L_2(y) \} \) \( / w = xy, x \in A, y \in B \} \leq L_3(w) \).

Therefore, \( L_3(w) = \{ L_1(x) \land L_2(y) \} \) \( / w = xy, x \in A, y \in B \} \).

Hence \( AB \) is recognizable set over \( \Sigma^* \), there exists an equivalent \( fpa \) \( M \) with fuzzy regular language \( L \) such that \( L = L_3 \).

**Theorem 3.6.**

Let \( A \subseteq \Sigma^* \) be a recognizable set with a fuzzy regular language \( L_1 \) which is accepted by an \( fpa \) \( M_1 \). Then \( A^* \) is recognizable with fuzzy regular language \( L \) accepted by an \( fpa \) \( M \) such that \( L = L_1^* \).

**Proof.**

Let \( M_1 = (Q_1, \Sigma, \mu_1, i_1, f_1) \) be an \( fpa \) with its fuzzy regular languages \( L_1 \).

Then \( L_1(x) > 0 \), for every \( x \in A \).

Define a fuzzy automaton \( M = (Q, \Sigma, \mu, i, f) \), where \( \mu: Q \times \Sigma \times Q \to [0,1] \) is defined as follows:

For every \( p, q \in Q_1 \), \( \mu(p, a, q) = \mu_1(p, a, q) \)

For every \( p, q \in Q_1 \) and if \( f_1(p) > 0 \) and \( i_1(q) > 0 \), \( \mu(p, \lambda, q) = f_1(p) \land i_1(q) \).

\( i: Q \to [0,1] \) is defined by \( i(p) = i_1(p) \), if \( p \in Q_1 \)

\( f: Q \to [0,1] \) is defined by \( f(p) = f_1(p) \), if \( p \in Q_1 \)

Hence \( \sum \mu(p, \lambda, q) = 1 \) and therefore \( M \) is the \( fpa \).

For \( L_2(w) = 0 \), the result is obvious.

Let \( w \in A^* \), \( L_2(w) > 0 \).

Then \( L_2(w) = \{ i(p) \land \mu^*(p, w, q) \land f(q) \} \) \( / q \in Q \} \) \( \land p \in Q \} \)

Since \( Q \) is finite and \( w \) is of finite length, we have

\[ L_2(w) = i(p_1) \land \mu(p_1, a_11, p_{11}) \land \ldots \land \mu(p_{1n_1}, a_{11}, p_{1n_1}) \land \mu(p_{1n_1}, \lambda, p_2) \]
\[ \land \mu(p_2, a_{21}, p_{21}) \land \ldots \land \mu(p_{2n_2}, a_{21}, p_{2n_2}) \land \mu(p_{2n_2}, \lambda, p_3) \]
\[ \land \ldots \ldots \]

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Since each term is non-negative, \( \mu(p_{ln_i}, \lambda, p_{i+1}) = f_1(p_{ln_i}) \land i_1(p_{i+1}), \; i = 1, 2, 3, ..., m \)

Let \( x_i \in A \).

Then \( L_1(x_i) > 0, \; i = 1, 2, 3, ..., m \).

Therefore, \( L_1(x_i) = V\{i_1(p) \land \mu_1(p, x_i, q) \land f_1(q)/q \in Q_1/p \in Q_1\} \)

Since \( Q_1 \) is finite and \( x_i = a_{i1} a_{i2} ... a_{in_i} \), we have

\( L_1(x_i) = i_1(p_{l_1}) \land \mu_1(p_{l_1}, a_{i1}, l_{i1}) \land ... \land \mu_1(p_{ln_i-1}, a_{in_i}, l_{in_i}) \land f_1(l_{in_i}) \)

where \( p_{l_i}, p_{l_j} \in Q_1, \; i = 1, 2, ..., n_i, \; i = 1, 2, 3, ..., m \)

Now, \( L_1(x_1) \land L_1(x_2) ... \land L_1(x_m) \)

\( = \left( i_1(p_{l_1}) \land \mu_1(p_{l_1}, a_{i1}, l_{i1}) \land ... \land \mu_1(p_{ln_i-1}, a_{in_i}, l_{in_i}) \land f_1(l_{in_i}) \right) \land ...

\( \land \left( i_1(p_{m}) \land \mu_1(p_{m}, a_{m1}, p_{m1}) \land ... \land \mu_1(p_{mn_{m-1}}, a_{mn_m}, p_{mn_m}) \land f_1(p_{mn_m}) \right) \)

Since each term is non-negative, \( f_1(p_{ln_i}) \land i_1(p_{i+1}) = \mu(p_{ln_i}, \lambda, p_{i+1}), \; i = 1, 2, 3, ..., m \).

Therefore, \( L_1(x_1) \land L_1(x_2) ... \land L_1(x_m) \)

\( = \left( i_1(p_{l_1}) \land \mu_1(p_{l_1}, a_{i1}, l_{i1}) \land ... \land \mu_1(p_{ln_i-1}, a_{in_i}, l_{in_i}) \land \mu(p_{ln_i}, \lambda, p_{p}) \land ...

\( \land \mu_1(p_{m}, a_{m1}, p_{m1}) \land ... \land \mu_1(p_{mn_{m-1}}, a_{mn_m}, p_{mn_m}) \land f_1(p_{mn_m}) \right) \)

$$\leq \bigvee \left\{ i(p_1) \land \mu(p_1, a_{11}, p_{11}) \land \ldots \land \mu(p_{1n_1-1}, a_{11}, p_{1n_1}) \land \mu(p_{1n_1}, \lambda, p_2) \right. $$

$$\land \mu(p_2, a_{21}, p_{21}) \land \ldots \land \mu(p_{2n_2-1}, a_{2n_2}, p_{2n_2}) \land \mu(p_{2n_2}, \lambda, p_3) \right.$$ 

$$\ldots \land \mu(p_m, a_{m1}, p_{m1}) \land \ldots \land \mu(p_{mn_m-1}, a_{mn_m}, p_{mn_m}) \land f(p_{mn_m}) \bigg\}$$

$$| p_i, p_{ij} \in Q, j = 1, 2, 3, \ldots, n_i, i = 1, 2, 3, \ldots, m$$

$$= L_2(a_{11} a_{12} \ldots a_{1n_1} a_{21} a_{22} \ldots a_{2n_2} \ldots a_{mn_m}) \land \mu(p_{1n_1}, \lambda, p_2) \land \mu(p_{2n_2}, \lambda, p_3) \right.$$ 

$$\ldots \land \mu(p_{mn_m}, \lambda, p_3) \bigg\}$$

$$L_1(x_1) \land L_1(x_2) \ldots \land L_1(x_m) \leq L_2(w), \text{ where } w = x_1 x_2 \ldots x_m$$

Hence

$$\forall \{ L_1(x_1) \land L_1(x_2) \ldots \land L_1(x_m) | w = x_1 x_2 \ldots x_m, each \ x_i \in A, i = 1, 2, 3, \ldots, m \ \} \leq L_2(w)$$

Therefore,

$$L_2(w) = \forall \{ L_1(x_1) \land L_1(x_2) \ldots \land L_1(x_m) | w = x_1 x_2 \ldots x_m, each \ x_i \in A, i = 1, 2, 3, \ldots, m \ \}$$

Whence $A^*$ is recognizable set over $\Sigma^*$, there exists an equivalent fpa $M$ with fuzzy regular language $L$ such that $L = L_2 = L_1^*$.

REFERENCES