Prime Labeling of Some 3-Regular Special Graphs

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ABSTRACT

A graph with vertex set $V$ is said to have a prime labeling if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for edge $xy$ the labels assigned to $x$ and $y$ are relatively prime. A graph which admits prime labeling is called a prime graph. In this paper, prime labeling for two special graphs namely Desargues graph and Heawood graphs are investigated.

Keywords: Prime labeling, Desargues graph, Heawood graph, Duplication.

1. INTRODUCTION

In this paper, only finite simple undirected graphs are considered. The graph $G$ has vertex set $V = V(G)$ and edge set $E = E(G)$. The set of vertices adjacent to a vertex $u$ of $G$ is denoted by $N(u)$. For notations and terminology we refer to Bondy and Murthy.

The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout. Two integers $a$ and $b$ are said to be relatively prime if their greatest common divisor is 1. Many researchers have studied prime graph. Fu.H has proved that the path $P_n$ on $n$ vertices is a prime graph. Deretsky et al. have proved that the cycle $C_n$ on $n$ vertices is a prime graph. Around 1980 Roger Entringer conjectured that all trees have prime labeling which has not been settled so far.

In S.K. Vaidhya and K.K. Kanmani have proved that the graphs obtained by identifying any two vertices duplicating arbitrary vertex and switching of any vertex in cycle $C_n$ admit prime labeling. In Meena and Vaithilingam have proved the Prime labeling for some helm related graphs.
**Definition 1.1**

Let $G = (V(G), E(G))$ be a graph with $p$ vertices. A bijection $f: V(G) \rightarrow \{1, 2, \ldots, p\}$ is called a prime labeling if for each edge $e = uv, \gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.

**Definition 1.2**

Duplication of a vertex $v$ of a graph $G$ produces a new graph $G'$ by adding a new vertex $v'$ such that $N(v') = N(v)$. In other words a vertex $v'$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ in $G$ are also adjacent to $v'$ in $G'$.

**Definition 1.3**

The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \bigcirc G_2$ formed by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 1.4**

A regular graph is a graph where every vertex has the same degree. A regular graph with vertices of degree $k$ is called a $k$-regular graph.

**Definition 1.5**

The Desargues Graph is a distance transitive Cubic graph with 20 vertices and 30 edges. The graph is as shown in the figure.
Definition 1.6
The Heawood graph is an undirected graph with 14 vertices and 21 edges as given in the adjoined figure. Heawood graph is a 3-regular graph.

2. MAIN RESULTS

Theorem 2.1
The Desargues Graph is a prime graph.

Proof:
Let \( G \) be a Desargues graph with 20 vertices and 30 edges. Let \( V(G) = \{v_1, v_2, \ldots, v_{20}\} \). Then \( |V(G)| = 20 \) and \( |E(G)| = 30 \).

The edge set \( E(G) \) is given by:

\[
E(G) = \{v_iv_{i+3} \mid 1 \leq i \leq 7\} \cup \{v_iv_{i+11} \mid 1 \leq i \leq 19\} \cup \{v_{11}v_{20}\} 
\]

Define a labeling \( f : V(G) \to \{1, 2, \ldots, 20\} \) by:

\[
f(v_1) = 9 \quad ; \quad f(v_2) = 7 \quad ; \quad f(v_3) = 1 \quad ; \quad f(v_4) = 3 \quad ; \quad f(v_5) = 5 \\
f(v_{2i}) = 2i, \quad i = 1, 2, 3, 4, 5 \quad ; \quad f(v_{i+1}) = i, \quad 11 \leq i \leq 20.
\]

Clearly, g.c.d \( \{f(v_i), f(v_{i+1})\} = 1 \), \( i = 11, 12, \ldots, 19 \)

\[
g.c.d \{f(v_i), f(v_{i+3})\} = 1, \quad i = 1, 2, 3, 4, 5, 6, 7
\]

\[
g.c.d \{f(v_{11}), f(v_{20})\} = 1
\]

\[
g.c.d \{f(v_i), f(v_{i+1})\} = 1, \quad i = 1, 2, \ldots, 7
\]

\[
g.c.d \{f(v_i), f(v_{i+3})\} = 1, \quad i = 8, 9, 10
\]
Then f admits prime labeling. Hence G is a prime graph.

**Illustration of theorem 2.1**

**Theorem 2.2**

Let G be a Desargues graph. Then the graph $G \Theta K_1$ is a prime graph.

**Proof:**

Let G be the Desargues graph. Let $G^* = G \Theta K_1$.

Let the vertices of $G^*$ be $\{v_1, v_2, \ldots, v_{20}, w_1, w_2, \ldots, \ldots, w_{10}\}$.

The edge set $E(G^*) = E(G) \cup \{v_1w_j \mid 11 \leq i \leq 20; 1 \leq j \leq 10\}$.

Then $|V(G^*)| = 30$ and $|E(G^*)| = 40$.

Define a labeling $f : V(G^*) \rightarrow \{1, 2, \ldots, \ldots, 30\}$ by

- $f(v_1) = 9; f(v_3) = 7; f(v_5) = 1; f(v_7) = 3; f(v_9) = 5$;
- $f(v_{2i}) = 2i, i = 1, 2, 3, 4, 5$;
- $f(v_i) = i, 11 \leq i \leq 20$;
- $f(\omega_i) = 20 + i, 1 \leq i \leq 10$.

Clearly all the vertices of $G^*$ have distinct labels and $\gcd \{f(x), f(y)\} = 1 \ \forall e = xy \in E(G^*)$. Then f admits a prime labeling. Hence $G^*$ is a prime graph.
Illustration of theorem 2.2

Figure:4   Prime labeling of $G \odot K_1$, $G$ is a Desargues graph

Theorem 2.3

Let $G$ be a Desargues graph. The graph obtained by duplication of a vertex of $G \odot K_1$ is a prime graph.

Proof:
Let $G'$ be the graph obtained by duplication of an arbitrary vertex by a new vertex $v_k$. Let the vertices of $G'$ be $\{v_1, v_2, \ldots, v_{20}, w_1, w_2, \ldots, w_{10}, v_k\}$.

Then $|V(G')| = 31$. Define a labeling $f : V(G') \to \{1, 2, \ldots, 31\}$ by

- $f(v_1) = 9$; $f(v_2) = 7$; $f(v_3) = 1$; $f(v_5) = 3$; $f(v_9) = 5$;
- $f(v_{2i}) = 2i$, $i = 1, 2, 3, 4, 5$; $f(v_i) = i$, $11 \leq i \leq 20$;
- $f(w_i) = 20 + i$, $1 \leq i \leq 10$; $f(v_k) = 31$.

Clearly g.c.d $\{f(x), f(y)\} = 1$ $\forall e = xy \in E(G')$.

Then $f$ admits a prime labeling. Hence the duplication of a vertex of $G \odot K_1$ is a prime graph.
Illustration of theorem 2.3

Theorem 2.4

Heawood graph is a prime graph.

Proof:

Let $G$ be the Heawood graph with 14 vertices and 21 edges. Then $|V(G)| = 14$ and $|E(G)| = 21$.

The edge set $E(G) = \{ v_i v_{i+1} / 1 \leq i \leq 13 \} \cup \{ v_1 v_{14} \} \cup \{ v_i v_{i+5} / i = 1,3,5,7,9 \}$

Define a labeling $f: V(G) \rightarrow \{1,2, \ldots, 14\}$

by $f(v_i) = i,$ $i = 2,3,4,6,7,8,9,10,11,12,13,14.$

$f(v_1) = 5 ; f(v_5) = 1$

Then $g.c.d \{ f(v_1), f(v_{i+1}) \} = 1, 1 \leq i \leq 13$

$g.c.d \{ f(v_i), f(v_{i+5}) \} = 1,$ $i = 1,3,5,7,9.$

$g.c.d \{ f(v_3), f(v_{13}) \} = 1$

$g.c.d \{ f(v_2), f(v_{11}) \} = 1 ; g.c.d \{ f(v_4), f(v_{13}) \} = 1.$

Thus $f$ admits a prime labeling. Hence $G$ is a prime graph.
Illustration of Theorem 2.4

**Theorem 2.5**

Let $G$ be the Heawood graph. The graph $G \circ K_1$ is a prime graph.

**Proof:**

Let $G' = G \circ K_1$. Let $V(G') = \{v_1, v_2, v_3, \ldots, v_{14}, w_1, w_2, \ldots, w_{14}\}$.

The edge set $E(G') = \{v_iv_{i+1} : 1 \leq i \leq 13\} \cup \{v_1v_{14}\} \cup \{v_iv_{i+5} : i = 1, 3, 5, 7, 9\}$

$\cup \{v_2v_{11}\} \cup \{v_4v_{13}\} \cup \{v_{i+1}w_i : i = 1 \leq i \leq 13\}$.

Define a labeling $f: V(G) \to \{1, 2, \ldots, 28\}$ by

$f(v_i) = i, i = 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14. ; f(v_2) = 5; f(v_3) = 1; f(w_i) = 14 + i, i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14; f(w_4) = 26; f(w_{12}) = 18$.

Clearly $\text{g.c.d}(f(x), f(y)) = 1 \forall e = xy \in E(G')$.

Thus $f$ admits a Prime labeling. Hence $G \circ K_1$ is a Prime Graph.

Illustration of Theorem 2.5

Figure 7: Prime labeling of $G \circ K_1$, $G$ is the Heawood graph
Theorem 2.6

Let G be the Heawood graph. Then duplication of a vertex of $G \sqcup K_1$ is a prime graph.

Proof:

Let G be the Heawood graph. Let $G^*$ be the graph obtained by duplication of a vertex of a graph $G \sqcup K_1$. Let the new vertex be $v_k$.

Let $V(G^*) = \{v_1, v_2, v_3, \ldots, v_{14}, w_1, w_2, \ldots, w_{14}, v_k\}$. Then $|V(G^*)|=29$

Define a labeling $f: V(G) \to \{1, 2, \ldots, 28\}$ by

- $f(v_i) = i, i = 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14$; $f(v_1) = 5$; $f(v_5) = 1$; $f(v_k) = 29$;
- $f(w_i) = 14 + i, i = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14$; $f(w_4) = 26$; $f(w_{12}) = 18$.

Clearly $g.c.d\{f(x), f(y)\} = 1 \forall e = xy \in E(G^*)$.

Thus $f$ admits a Prime labeling. Hence the duplication of a vertex of $G \sqcup K_1$ is a Prime Graph.

Illustration of Theorem 2.6

![Diagram](image_url)

Figure: 8 prime labeling for duplication of a vertex $v_5$ of $G \sqcup K_1$

3. CONCLUSION

In this paper, prime labeling for the special graphs namely Desargues graph and Heawood graph have investigated. Extending the graphs by the operation duplication is also discussed. Further discussion will be performed in this context.
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