Common Fixed Point Theorems Using Implicit Relation and Property (JCLR) in Fuzzy Metric Spaces

M.S. CHAUHAN$^1$, DHEERAJ AHEERE$^2$ and BHARAT SINGH$^3$

$^1$Rajabhoj Govt. College, Mandideep Dist., Raisen, M.P., INDIA.
$^2$Research Scholar, Vikram University, Ujjain, M.P. INDIA.
$^3$SOC & E IPS Academy Indore, M.P., INDIA.

(Received on: August 14, 2013)

ABSTRACT

The aim of this paper is to prove a common fixed point theorem for six maps via notion of pairwise commuting maps in fuzzy metric space satisfying contractive type implicit relation. Our main result extends the result of Aalam, Kumar and Pant$^8$ and Suneel Kumar, Sunny Chauhan$^{19}$ and Sunny Chauhan, et al.$^{18}$.

Keywords: Fuzzy metric space, Implicit relation, Weakly compatible maps, Property (CLRg), Property (JCLR) and fixed point.

I. INTRODUCTION

For the last quarter of the twentieth Century, there has been considerable interest to study the common fixed point of commuting maps and its weaker forms. Mishra et al.$^{17}$ extended the notion of compatible maps (introduction by Jugek$^5$ in metric space) under the name of asymptotically commuting maps and B. Singh and Jain$^3$ extended the notion of weakly compatible maps (introduction by Jugek$^5$ in metric space) to FM-space. V. Pant and Pant$^{19}$ extended the study of common fixed point of a pair of non-compatible maps studied by Pant$^{19}$ in metric space.

Zadeh$^{11}$ introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek$^9$ introduced the concept of fuzzy metric space (FM-spaces), which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to FM-spaces by various authors viz George and Veeramani$^1$, Grabiec$^{15}$ and others.

Implicit relations and CLRg property are used as a tool for finding common fixed point of contraction maps.
Recently, Aalam, Kumar and Pant proved a common fixed point theorem without completeness of space and continuity of involved mappings in FM-space, which generalizes the results of Singh and Jain. In this paper, we prove a common fixed point theorem for six-maps in FM-space satisfying contractive type implicit relations.

II. PRELIMINARIES

Definition 2.1
Let X be any set. A fuzzy set M in X is a function with domain X and values in [0, 1].

Definition 2.2
A binary operation * : [0, 1] x [0, 1] → [0, 1] is called a continuous t-norm if ( [0,1],*) is an abelian topological monoid with the unit 1 such that a∗b ≤ c∗d whenever a ≤ c and b ≤ d. ∀ a,b,c,d ∈ [0,1].

Definition 2.3
The triplet (X, M, *) is a FM-space if X is an arbitrary set, M is a continuous t-norm and M is a fuzzy set in X x [0, ∞) satisfying the following conditions for all x,y,z ∈ X and t,s >0,
[1] M(x,y,t) = 1 for all t >0 if and only if x = y;
[2] M(x,y,0) = 0;
[3] M(x,y,t) = M(y,x,t);
[4] M(x,y,t)*M(y,z,s) ≤ M(x,z,t+s);
[5] M(x,y,·) : [0, ∞) → [0, 1] is left continuous.

Example 2.1
Let (X,d) be a metric space. Define a∗b = ab(or a∗b = min {a,b}) for all x,y ∈ X and t >0,
M(x,y,t) = \frac{t}{t+d(x,y)}.
Then (X,M,*) is a FM-space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.

Lemma 2.1
Let M(x,y,·) be a FM-space. Then M is a continuous function on X² x (0,∞).

Definition 2.4
Let A and S maps from a FM-space (X,M,*) into itself. The maps A and S are said to be compatible (or asymptotically commuting), if for all t,
limit_{n→∞} M(ASx_n, SAx_n, t) = 1.
Whenever \{x_n\} is a sequence in X such that limit_{n→∞} Ax_n = z = limit_{n→∞} Sx_n for some z ∈ X.

Definition 2.5
Let A and S be maps from a FM-space (X,M,*) into itself.
The maps are said to be weakly compatible if they commute at their coincidence points, i.e.
Az = Sz implies that ASz = SAz.

Definition 2.6
A pair (f,g) of self mappings of a fuzzy metric space (X,M,*) is said to satisfy the “common limit in the range of g” property (CLRg) if there exists a sequence \{x_n\} in X such that limit_{n→∞} fx_n = limit_{n→∞} gx_n = u, for some u ∈ X.

Example 2.2 Let \((X, M, *)\) be a fuzzy metric space with \(X = [0, \infty)\) and

\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|}, & \text{if } t > 0; \forall x, y \\
0, & \text{if } t > 0.
\end{cases}
\]

Define self mappings \(f\) and \(g\) on \(X\) by \(f(x) = x + 3\) and \(g(x) = 4x\), \(\forall x \in X\). Let a sequence \(\{x_n\}\) be \(1 + \frac{1}{n}\) \(\forall n \in \mathbb{N}\) in \(X\) we have,

\[
l_{\lim n \to \infty} f x_n = l_{\lim n \to \infty} g x_n = 4 = g(1) \in X,
\]

which shows that \(f\) and \(g\) satisfy the (CLRg) property.

Definition 2.8\(^{18}\) Let \((X, M, *)\) be a fuzzy metric space and \(A, SR, B, TH: X \to X\). The pair \((A, SR)\) and \((B, TH)\) are said to satisfy the “joint common limit in the range” property (JCLR property) if there exists a sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
l_{\lim n \to \infty} Ax_n = l_{\lim n \to \infty} SR x_n = l_{\lim n \to \infty} By_n = l_{\lim n \to \infty} TH y_n = SR u = TH u,\] for some \(u \in X\).

III. MAIN RESULTS

Theorem 3.1 Let \(A, B, R, S, H\) and \(T\) be self maps of a FM-space \((X, M, *)\) satisfying

(3.1.1) \((A, SR)\) or \((B, TH)\) satisfies the property (JCLR);

(3.1.2) \(\phi \left( \frac{M(Ax, By, kt), 1 + M(SRx, Thy, t)}{M(Ax, SRx, t), M(By, Thy, t)} \right) \geq 0;\) for all \(t > 0, x, y \in X\) and for some \(\phi \in \Phi\);

(3.1.3) \(A(X) \subseteq TH(X), B(X) \subseteq SR(X)\);

(3.1.4) One of \(A(X), B(X), SR(X)\) and \(TH(X)\) is a complete subspace of \(X\).

Then the pairs \((A, SR)\) and \((B, TH)\) have a point of coincidence each. Moreover, \(A, B, R, S, H\) and \(T\) have a unique common fixed point provided the pairs \((A, SR)\) and \((B, TH)\) commute pairwise. (i.e. \(AS = SA, AR = RA, SR = RS, BT = TB, BH = HB\) and \(TH = HT\)).

Proof: Since the pair \((A, SR)\) and \((B, TH)\) satisfies the property (JCLR) , then there exists a sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
l_{\lim n \to \infty} Ax_n = l_{\lim n \to \infty} SR x_n = l_{\lim n \to \infty} By_n = l_{\lim n \to \infty} TH y_n = SR u = TH u,\] for some \(u \in X\).

Now we show that \(TH u = Bu\). Using (3.1.2), with \(Put = x_n\) and \(y = u\).

\[
\Rightarrow \Phi \left( \frac{M(Ax_n, Bu, kt), 1 + M(SRx_n, Thu, t)}{M(Ax_n, SRx_n, t), M(By, Thu, t)} \right) \geq 0,
\]

\[
\Rightarrow \Phi \left( \frac{M(Ax_n, Bu, t), M(SRx_n, Thu, t)}{M(Ax_n, SRx_n, t), M(By, Thu, t)} \right) \geq 0,
\]

\[
\Rightarrow \Phi \left( \frac{M(TH u, Bu, t)}{M(TH u, Thu, t)} \right) \geq 0,
\]

Since \(\Phi\) is non-increasing in 2nd and 3rd argument.

\[
\Rightarrow \Phi \left( \frac{M(TH u, Bu, kt), 1, t)}{M(TH u, Thu, t)} \right) \geq 0,
\]

Therefore by implicit relation.
\[ M(THu, Bu, kt) \geq M(THu, Bu, t) \]

By lemma 2.2 we have
\[ THu = Bu. \]

Next we show that Au = THu. Using (3.1.2), with \( x = u \) and \( y = y_n \)
\[ \Rightarrow \phi \left( M(AuByu,kt), 1+M(SRu,THyn,t), M(Au,SRu,t), M(Byu,THyn,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(AuByu,kt), M(SRu,THyn,t), M(Au,SRu,t), M(Byu,THyn,t) \right) \geq 0, \]

By JCLR property
\[ \Rightarrow \phi \left( M(Au,THu,kt), M(THu,THu,t), M(Au,THu,t), M(THu,THu,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(Au,THu,kt), M(Au,THu,t), M(THu,THu,t) \right) \geq 0, \]
Since \( \phi \) is non-increasing in 2\(^{nd}\) and 4\(^{th}\) argument
\[ \Rightarrow \phi \left( M(Au,THu,kt), M(Au,THu,t), M(THu,THu,t) \right) \geq 0, \]

Therefore by implicit relation
\[ M(Au,THu,kt) \geq M(Au,THu,t) \]
By lemma 2.2 we have
\[ \Rightarrow Au = THu. \]

Now assume that \( z = Au = Bu = THu = SRu \).
Since the pair (A,SR) is weak compatible
\[ \Rightarrow Az = SRz. \]
Similarly the pair (B,TH) is also weak compatible then,
\[ BTHu = THBu \Rightarrow Bz = THz. \]

Now we have to show that \( Az = z \). Using (3.1.2), with \( x = z \) and \( y = u \)
\[ \Rightarrow \phi \left( M(Az,Bu,kt), 1+M(SRz,THu,t), M(Az,SRz,t), M(Bu,THu,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(Az,Bu,kt), M(SRz,THu,t), M(Az,SRz,t), M(Bu,THu,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(Az,Bu,kt), M(Az,Bu,t), M(Bu,THu,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(Az,Bu,kt), M(Az,Bu,t), M(Bu,THu,t) \right) \geq 0, \]
Since \( \phi \) is non-increasing in 3\(^{rd}\) and 4\(^{th}\) argument
\[ \phi \left( M(Az,Bu,kt), M(Az,Bu,t), M(Bu,THu,t) \right) \geq 0, \]
Therefore by implicit relation
\[ M(Az,z,kt) \geq M(Az,z,t) \]
By lemma 2.2 we have
\[ \Rightarrow Az = z, \Rightarrow z = Az = SRz. \]

Now we have to show that \( Bz = z \). Using (3.1.2), with \( x = u \) and \( y = z \)
\[ \Rightarrow \phi \left( M(AuBz,kt), 1+M(SRu,THx,t), M(Au,SRu,t), M(Bz,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(AuBz,kt), M(SRu,THx,t), M(Au,SRu,t), M(Bz,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(z,Bz,kt), M(z,Bz,t), M(Bz,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(z,Bz,kt), M(z,Bz,t), M(Bz,THx,t) \right) \geq 0, \]
Since \( \phi \) is non-increasing in 3\(^{rd}\) and 4\(^{th}\) argument
\[ \Rightarrow M(z,z,kt) \geq M(z,z,t) \]
By lemma 2.2 we have
\[ \Rightarrow z = Bz. \]
Therefore we have \( z = Az = Bz = SRz = THz. \)
\[ \Rightarrow A, SR, B \text{ and } TH \text{ have common fixed point is a point } z. \]

**For uniqueness:** Let \( w \) be another common fixed point of the mappings \( A, SR, B \) and \( TH \)

Using (3.1.2), with \( x = z \) and \( y = w \)
\[ \Rightarrow \phi \left( M(Az,Bo,kt), 1+M(SRz,THx,t), M(Az,SRz,t), M(Bo,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(Az,Bo,kt), M(SRz,THx,t), M(Az,SRz,t), M(Bo,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(z,w,kt), M(z,w,t), M(w,THx,t) \right) \geq 0, \]
\[ \Rightarrow \phi \left( M(z,w,kt), M(z,w,t), M(w,THx,t) \right) \geq 0, \]
Since \( \phi \) is non-increasing in 3\(^{rd}\) and 4\(^{th}\) argument
\[ \Rightarrow \phi \left( M(z,w,kt), M(z,w,t), M(w,THx,t) \right) \geq 0, \]
Therefore by implicit relation
\[ M(z,w,kt) \geq M(z,w,t) \]
By lemma 2.2 we have
\[ z = w. \]

Therefore \( A, SR, B \) and \( TH \) have a unique a common fixed point.

We can also prove the same result if the pair \( (A, S) \) satisfies the property \((JCLR)\). The proof is similar when \( TH(X) \) is assumed to be a complete subspace of \( X \). The remaining two cases pertain essentially to the previous cases. If we assume that \( A(X) \) is a complete subspace of \( X \), then \( z \in A(X) \subseteq TH(X) \) or \( B(X) \) is a complete subspace of \( X \), then \( z \in B(X) \subseteq SR(X) \).

Thus we can establish that both the pairs \( (A, SR) \) and \( (B, TH) \) have a point of coincidence each. This completes proof.

**Corollary 3.2** (Theorem 3.1) Let \( A, B, S \) and \( T \) be self maps of an FM-space \((X,M,\ast)\) satisfying:

1. \((A,S)\) or \((B,T)\) satisfies the property \((JCLR)\),
2. \( \phi \left( M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t) \right) \geq 0 \), for all \( t > 0, x, y \in X \) and for some \( \phi \in \Phi \),
3. \( A(X) \subseteq T(X) \), \( B(X) \subseteq S(X) \),
4. One of \( A(X), B(X), S(X) \) and \( T(X) \) is a complete subspace of \( X \).

Then the pairs \( (A, S) \) and \( (B, T) \) have a point of coincidence each. Moreover, \( A, B, S \) and \( T \) have a unique common fixed point provided both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

**Corollary 3.3** Let \( A, S \) be self maps of an FM-space \((X,M,\ast)\) satisfying:

1. \((A,S)\) satisfies the property \((JCLR)\),
2. \( \phi \left( M(Ax, Ay, t), M(Sx, Sy, t), M(Ax, Sx, t), M(Ay, Sy, t) \right) \geq 0 \), for all \( t > 0, x, y \in X \) and for some \( \phi \in \Phi \),
3. \( A(X) \subseteq S(X) \),
4. One of \( A(X) \) and \( S(X) \) is a complete subspace of \( X \).

Then the pairs \( (A, S) \) has a point of coincidence. Moreover, \( A \) and \( S \) have a unique common fixed point provided the pairs \((A, S)\) is weakly compatible.

**ACKNOWLEDGEMENTS**

We are grateful to the Professor Bijendra Singh, Professor & Head V. H. Badshah and Reader, S. K. Tiwari, School of Studies in Mathematics Vikram University, Ujjain (M.P.) India, for their cooperation and valuable suggestions in the preparation of this paper.

**REFERENCES**


14. Manish Jain and Sanjay Kumar, A Common fixed point Theorem in Fuzzy Metric Space using the property (CLRg), *Journal Of Mathematics*, VOLUME X, NUMBER X-XX-XX (20XX).


