Invariance Under Cordiality of Path Union of Crown Graph

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ABSTRACT

In a crown graph $C_n^t$ has a pendent edge fused to each vertex of $C_n$. We obtain a $t$-crown graph of $C_4$ denoted by $G = \text{C}_4^t$ by fusing $t$ - pendent edges to each of $C_4$ vertex. We show that path union $P_m(G)$ is cordial for all $t$ and all $m$. Further if we change the vertex on $G$ to obtain path union so as to obtain a non-isomorphic structure (there are only two such structures possible) we still get cordial labeling.

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2. INTRODUCTION

The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Holton\textsuperscript{5} Graph Theory by Harary\textsuperscript{6}, A dynamic survey of graph labeling by J.Gallian\textsuperscript{8} and Douglas West\textsuperscript{9}. I.Cahit introduced the concept of cordial labeling\textsuperscript{4}. $f:V(G)\rightarrow \{0,1\}$ be a function. From this label of any edge $(uv)$ is given by $|f(u)-f(v)|$. Further number of vertices labeled with 0 i.e. $v(f(0))$ and the number of vertices labeled with 1 i.e. $v(f(1))$ differ at most by one. Similarly number of edges labeled with 0 i.e. $e(f(0))$ and number of edges labeled with 1 i.e. $e(f(1))$ differ by at most one. Then the function $f$ is called as cordial labeling. Cahit has shown that: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all $m$ and $n$; the friendship graph $C_3^0$ (i.e., the one-point union of $t$ copies of $C_3$) is cordial if and only if $t$ is not congruent to 2 (mod 4); all fans are cordial; the wheel $W_n$ is cordial if and only if $n$ is not congruent to 3 (mod 4). A lot of work has been done in this type of labeling. One may refer dynamic survey by J. Gallian\textsuperscript{8}.

Our focus of attention is on path unions on different graphs. For a given graph there are different path unions (upto isomorphism) structures possible. It depends on which point on
G is used to fuse to obtain path union. We have shown that for \( G = \text{bull on } C_3 \), \( \text{bull on } C_4, C_5^+, C_6^+ - e \) the different path union \( P_m(G) \) are cordial. It is called as invariance under cordial labeling. We use the convention that \( v_f(0,1) = (a,b) \) to indicate the number of vertices labeled with 0 are \( a \) in number and that number of vertices labeled with 1 are \( b \). Further \( e_f(0,1) = (x,y) \) we mean the number of edges labeled with 0 are \( x \) and number of edges labeled with 1 are \( y \). The graph whose cordial labeling is available is called as cordial graph. In this paper we define crown \( C_{n+t} \) graph and obtain path union graphs on it. (for given \( t \) and \( n = 3,4 \))

3. PRELIMINARIES

3.1 Fusion of vertices. Let \( u \neq v \) be any two vertices of \( G \). We replace these two vertices by a single vertex say \( x \) and all edges incident to \( u \) and \( v \) are now incident to \( x \). If loop is formed then it is deleted.

3.2 Crown graph. This concept was initially defined for \( C_n \) and denoted by \( C_n^+ \). It is obtained by fusing an edge to each vertex of \( C_n \). We obtain \( C_{n+t} \) by fusing equal number say \( t \) pendent edges at each vertex of \( C_n \). Thus \( C_{n+t} \) has \( n(1+t) \) vertices and as much edges.

3.3 Path union of \( G \) i.e. \( P_m(G) \) is obtained by taking a path \( P_m \) and \( m \) copies of graph \( G \). Fuse a copy each of \( G \) at every vertex of path at given fixed point on \( G \). It has \( mp \) vertices and \( mq + m - 1 \) edges, where \( G \) is a \((p, q)\) graph. If we change the vertex on \( G \) that is fused with vertex of \( P_m \) then we generally get a path union non-isomorphic to earlier structure.

4. MAIN RESULTS

Theorem 4.1 Let \( G = C_4^+ \). A path union obtained by fusing a pendent vertex of a copy of \( G \) at each vertex of path \( P_m \) is cordial.

Proof: Let a \( C_4 \) cycle in terms of vertices be given by \((a, b, c, d, a)\). The \( t \)-pendent edges adjacent at vertex \( a \) be given by \( a_1, a_2, \ldots, a_t \). Similarly \( b_1, b_2, \ldots, b_t \) be adjacent edges to vertex \( b \), and \( c_1, c_2, c_3, \ldots, c_t \) are adjacent edges at vertex \( c \) and \( d_1, d_2, \ldots, d_t \) are edges adjacent to vertex \( d \) of \( C_4 \). To obtain a labeled copy of \( G \) define a function \( f: V(G) \rightarrow \{0,1\} \) as follows:

\[
\begin{align*}
f(a) &= 1; \\
f(b) &= 0; \\
f(c) &= 0; \\
f(d) &= 1; \\
\end{align*}
\]

Further the pendent edges are labeled as follows:

\[
\begin{align*}
f(a_i) &= 1 \text{ for all } i = 1, 2, \ldots, t; \\
f(b_i) &= 0 \text{ for all } i = 1, 2, \ldots, t; \\
f(c_i) &= 1 \text{ for all } i = 1, 2, \ldots, t; \\
f(d_i) &= 0 \text{ for all } i = 1, 2, \ldots, t; \\
\end{align*}
\]

For a labeled copy of \( G \) we have label distribution given by \( v_f(0,1) = e_f(0,1) = (t+2, t+2) \).
We extend the same function $f$ above to apply on $P_m(G)$ by $f: V(P_m(G)) \rightarrow \{0,1\}$.

**Type I path union**

Let the path be given by $P_m = x_1, x_2, \ldots, x_m$. Fuse a copy of $G$ at pendent vertex with label 1 at $x_i$ if $i \equiv 1, 2 \pmod{4}$, the label of this path vertex will be 1, and with label 0 if $i \equiv 3, 4 \pmod{4}$, the label of this path vertex will be 0. Actually a sequence of vertex labels 1, 1, 0, 0 will be repeated on the vertex label of $P_m$.

The label number distribution is given for $m$ a even number given by $m = 2x$, $x = 0, 1, \ldots$, we have $v_f(0,1) = (m(t+2),m(t+2))$. $e_f(0,1)= (m(t+2)+\frac{m}{2},m(t+2)+\frac{m}{2}-1)$.

When $m$ is an odd number given by $m = 2x+1$, $x = 0, 1, \ldots$ we have label number distribution given by we have $v_f(0,1) = (m(t+2),m(t+2))$. $e_f(0,1)= (m(t+2)+\frac{m-1}{2},m(t+2)+\frac{m-1}{2})$. Thus the graph is cordial.

**Theorem 4.2** Let $G = C_4^+$. A path union obtained by fusing a cycle $C_4$ vertex of a copy of $G$ at each vertex of path $P_m$ is cordial.

**Proof:** Let a $C_4$ cycle in terms of vertices be given by $(a, b, c, d)$. The t-pendent edges adjacent at vertex $a$ be given by $a_1, a_2, \ldots, a_t$. Similarly $b_1, b_2, \ldots, b_t$ be adjacent pendent edges to vertex $b$, and $c_1, c_2, c_3, \ldots, c_t$ are adjacent pendent edges at vertex $c$ and $d_1, d_2, \ldots, d_t$ are pendent edges adjacent to vertex $d$ of $C_4$.

To obtain a labeled copy of $G$ define a function $f: V(G) \rightarrow \{0,1\}$ as follows:

- $f(a) = 1$;
- $f(b) = 0$;
- $f(c) = 0$;
- $f(d) = 1$;

Further the pendent edges are labeled as follows:
f(a_i)=1 for all i = 1, 2, ..t;
f(b_i) =0 for all i = 1, 2, .., t;
f(c_i) = 1 for all i = 1, 2, .., t;
f(d_i)= 0  for all i = 1, 2, .., t;
For a labeled copy of G we have label distribution given by \( v_f(0,1) = e_f(0,1) = (t+2,t+2). \)

We extend the same function \( f \) above to apply on \( P_m(G) \) by \( f: V(P_m(G)) \rightarrow \{0,1\}. \)

**Type II path union**

Let the path be given by \( P_m = x_1, x_2, \ldots, x_m. \) Fuse a copy of G at t+2 degree vertex of \( C_4 \) with label 1 at \( x_i \) if \( i \equiv 1, 2 \) (mod 4), the label of this path vertex will be 1, and with label 0 if \( i \equiv 3, 4 \) (mod 4), the label of this path vertex will be 0. Actually a sequence of vertex labels 1, 1, 0, 0 will be repeated on the vertex label of \( P_m. \) The figure below explains the thing when \( t = 3 \) and \( m = 4. \)

![Fig 4.1 P⁴(C₄) with cycle vertex whose degree is t+2 is fused path vertex to form path union.](image)

The label number distribution is given for \( m \) a even number given by \( m = 2x, x = 0, 1, \ldots \), we have \( v_f(0,1) = e_f(0,1) = (m(t+2),m(t+2)). \)

When \( m \) is an odd number given by \( m = 2x+1, x = 0, 1, \ldots \) we have label number distribution given by we have \( v_f(0,1) = (m(t+2),m(t+2)). \)

Thus the graph is cordial.

**Theorem 4.2** Let \( G = C_3^{e_1}. \) A path union obtained by fusing a pendent vertex of a copy of \( G \) at each vertex of path \( P_m \) is cordial.

**Proof:** Let a \( C_3 \) cycle in terms of vertices be given by \( (a, b, c, a). \) The t-ependent edges adjacent at vertex \( a \) be given by \( a_1, a_2, \ldots, a_t. \) Similarly \( b_1, b_2, \ldots, b_t \) be adjacent pendent edges to vertex \( b, \) and \( c_1, c_2, c_3, \ldots, c_t \) are adjacent pendent edges at vertex \( c \) of \( C_3. \)

To obtain a labeled copy of \( G \) define a function \( f: V(G) \rightarrow \{0,1\} \) as follows:

\[
\begin{align*}
f(a) &= 1; \\
f(b) &= 0; \\
f(c) &= 0;
\end{align*}
\]

We consider two cases on \( t \) as follows:
Case t is odd number given by \( t = 2x+1, x = 0, 1, 2, \ldots \)
\( f(a_i) = 1 \) for \( i= 1, 2, \ldots, x, x+1; \)
\( f(a_i) = 0 \) for \( i= x+2, x+3, \ldots, 2x+1 \)
\( f(b_i) = 0 \) for all \( i = x+2, x+3, \ldots, 2x+1; \)
\( f(c_i) = 1 \) for all \( i = 1, 2, \ldots, 2x+1. \)

For \( t = 1 \) diagram below explains the required labeling.

For \( t > 1, t = 2x+1 \). The diagram explains the labeling.

We extend the same function \( f \) above to apply on \( P_m(G) \) by \( f: V(P_m(G)) \rightarrow \{0,1\} \).

Let the path be given by \( P_m = x_1, x_2, \ldots, x_m \). Fuse a copy of \( G \) pendent vertex of \( G \) with label 1 at \( x_i \) if \( i \equiv 1, 2 \mod 4 \), the label of these path vertex will be 1, and with label 0 if \( i \equiv 3, 4 \mod 4 \), the label of this path vertex will be 0. Actually a sequence of vertex labels 1, 1, 0, 0 will be repeated on the vertex label of \( P_m \). The label number distribution is given for \( m \) a even \( m = 2y, y= 1, 2, \ldots \), number is given by we have \( v_f(0,1) = (m(3x+3),m(3x+3)). \)
\( e_f(0,1) = (m(3x+3)+\frac{m}{2},m(3x+3)+\frac{m}{2}-1). \)

The label number distribution is given for \( m \) a even \( m = 2y+1, y= 0, 1, \ldots \), number is given by we have \( v_f(0,1) = (m(3x+3),m(3x+3)). \)
\( e_f(0,1) = (m(3x+3)+\frac{m-1}{2},m(3x+3)+\frac{m-1}{2}). \)
Let the path be given by \( P_m = x_1, x_2, \ldots, x_m \). Fuse a copy of \( G \) at \( t+2 \) degree vertex of \( C_3 \) with label 1 at \( x_i \) if \( i \equiv 1, 2 \pmod{4} \), the label of this path vertex will be 1, and with label 0 if \( i \equiv 3, 4 \pmod{4} \), the label of this path vertex will be 0. Actually a sequence of vertex labels 1, 1, 0, 0 will be repeated on the vertex label of \( P_m \). The label number distribution is same as above. Case \( t = 2x \).

We give two type of labels.

**Type A label:**
- \( f(a_i) = 1 \) for \( i = 1, 2, \ldots, x \);
- \( f(a_i) = 0 \) for \( i = x+1, x+2, x+3, \ldots, 2x \);
- \( f(b_i) = 0 \) for all \( i = 1, 2, \ldots, 2x \);
- \( f(c_i) = 1 \) for all \( i = 1, 2, \ldots, 2x \).

Thus type A label has label number distribution given by \( v_f(0,1) = (3x+2,3x+1) \) and \( e_f(0,1) = (3x+1,3x+2) \).

**Type B label:**
- \( f(a_i) = 1 \) for \( i = 1, 2, \ldots, x+1 \);
- \( f(a_i) = 0 \) for \( i = x+2, x+3, \ldots, 2x \);
- \( f(b_i) = 0 \) for all \( i = 1, 2, \ldots, 2x \);
- \( f(c_i) = 1 \) for all \( i = 1, 2, \ldots, 2x \).

Thus type B label has label number distribution given by \( v_f(0,1) = (3x+1,3x+2) \) and \( e_f(0,1) = (3x+2,3x+1) \).

To obtain a labeled copy of path union we extend the same function \( f \) above to apply on \( P_m(G) \) by \( f: V(P_m(G)) \rightarrow \{0,1\} \).

**Type I path union**

Let the path be given by \( P_m = x_1, x_2, \ldots, x_m \). Fuse a copy of \( G \) pendent vertex of type A with label 1 at \( x_i \) if \( i \equiv 1, 2 \pmod{4} \), the label of these path vertex will be 1, and type B with label 0 if \( i \equiv 3, 4 \pmod{4} \), the label of this path vertex will be 0.

When \( m = 1 \) use type A label.

Actually a sequence of vertex labels 1, 1, 0, 0 will be repeated on the vertex label of \( P_m \). The label number distribution is given for \( m \) a even \( m = 2y \), \( y = 1, 2, \ldots \), by we have \( v_f(0,1) = (y(6x+3),m(6x+3)) \) and \( e_f(0,1) = (m(6x+3)+\frac{m}{2},m(6x+3)+\frac{m}{2}-1) \). When \( m \) is an odd number say \( m = 2y+1 \) first obtain labeled copy for \( m = 2y \) from one end of path \( P_m \). Label the copy of \( G \) at last vertex of path with type A label. The label of vertex will be same as the label of pendent vertex. This should be ‘0’ if \( 2y+1 \equiv 3 \pmod{4} \), and ‘1’ if \( 2y+1 \equiv 1 \pmod{5} \). The label number
distribution is the label number distribution is given for m a even m = 2y+1, y= 0, 1... , number is given by we have \( v_i(0,1) = (m(3x+3)+3x+2, m(3x+3)+3x+1). e_i(0,1) = (m(3x+3)+3x+1+\frac{m-1}{2}, m(3x+3)+3x+2+\frac{m-1}{2}) \).

Thus the graph is cordial.

It follows that there are only two isomorphic structures possible on path union. One is if we use any of pendent vertex as a vertex to fuse with path vertex and the other when we use any triangle vertex (a, b or c) to obtain path union.

Thus the graph is cordial.

CONCLUSIONS

In this paper we have used the \( C_n^m (n = 3, 4) \) graph to obtain path union on it. There are two structures of path union possible depending on if we use a cycle vertex or the pendent vertex to obtain path union. We have shown that in both cases th graph obtained is cordial. Thus though the structure is changed (keeping the structure as path union as fixed) the resultant graph is cordial. This may be referred as invariance of path union of \( C_n^m (n = 3, 4) \) under cordiality. It is necessary to investigate the cordiality for given every n and any t.

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