Generalized Polygonal Sum Labeling of Graphs

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(Received on: June 15, 2018)

ABSTRACT

Let $G$ be a $(p,q)$ graph. A graph $G$ admits Polygonal sum labeling if a one to one function $f : V(G) \rightarrow N$ (where $N$ is a set of all non-negative integers) that induces a bijection $f^+ : E(G) \rightarrow \{P_k(1), P_k(2), \ldots, P_k(q)\}$ of the edges of $G$ defined by $f^+(uv) = f(u) + f(v)$ for every $e = uv \in E(G)$ where $P_k(1), P_k(2), \ldots, P_k(q)$, where $k \geq 3$ are the first $q$ polygonal numbers. A graph which admits such labeling is called Polygonal sum graph. In this paper we characterize some families of graphs such as Path, Comb, Star graph, Subdivision of star, Bistar, $S_{m,n,r}$ and Coconut tree admit polygonal sum labeling. This work is a nice composition of graph theory and combinatorial number theory.

Mathematical Subject code : 05C78

Keywords: Polygonal numbers, Polygonal sum labeling, Polygonal sum graph.

1. INTRODUCTION

In this paper, we consider non-trivial finite, simple undirected graph. The set of vertices and edges of a graph $G (p, q)$ will be denoted by $V(G)$ and $E(G)$ respectively. The various graph theoretic notations and terminology we follow Frank Harary2 and for number theory we follow Burton1. Many kind of labeling have been, defined and studied by many authors and an excellent survey of graph labelings can be found in3.

Hedge and Shankaran4 introduced triangular sum labeling. A graph is said to admit a triangular sum labeling, if its vertices can be labeled by non-negative integers so that the values on the edges, obtained as the sum of the labels of their end vertices, are the first $q$ triangular numbers.
A pentagonal sum labeling was introduced by S. Murugesan et al. A pentagonal sum labeling is an injection \( f: V(G) \to \mathbb{N} \) that induces a bijection \( f^+: E(G) \to \{A_1, A_2, \ldots, A_q\} \) of the edges of \( G \) defined by \( f^+(uv) = f(u) + f(v) \) for every \( e = uv \in E(G) \). Where \( A_1, A_2, \ldots, A_q \) are the first \( q \) pentagonal numbers. A graph which admits such labeling is called pentagonal sum graph.

In this paper, we define generalized polygonal sum labeling and prove some families of graphs such as Path, Comb, Star graph, Subdivision of star, Bistar, \( S_{m,n,r} \) and Coconut tree admit centered polygonal sum labeling.

**Definition 1.1**

Polygons are just numbers that create certain polygon. The generalized polygonal numbers are denoted by \( P_k(n) \), then \( P_k(n) = \frac{n}{2} [(k - 2) n - (k - 4)] \), where \( k \geq 3 \) is the number sides of the polygon. For \( k = 3 \) it gives triangular numbers, For \( k = 4 \) it gives tetragonal numbers and so on.

**Definition 1.2**

A polygonal sum labeling of a graph \( G \) is a one to one function \( f: V(G) \to \mathbb{N} \) (where \( \mathbb{N} \) is set of all non-negative integers) that induces a bijection \( f^+: E(G) \to \{P_3(1), P_3(2), \ldots, P_3(q)\} \) of the edges of \( G \) defined by \( f^+(uv) = f(u) + f(v) \) for every \( e = uv \in E(G) \). The graph which admits such labeling is called polygonal sum graph.

For \( k = 3 \) the above labeling gives triangular sum labeling, For \( k = 4 \) it gives tetragonal sum labeling and so on.

**2. MAIN RESULTS**

**Theorem 2.1** : The path \( P_n \) admits polygonal sum labeling.

**Proof** :
Let \( P_n : u_1u_2 \ldots u_n \) be a path and \( v_i = u_iu_{i+1} (1 \leq i \leq n-1) \) be the edges.
For \( i = 1, 2, 3, \ldots, n \), define

\[
f(u_i) = \begin{cases} 
\frac{1}{4}(i - 1)[(k - 2)i - (k - 4)] & \text{if } i \text{ is odd} \\
\frac{1}{4}i[(k - 2)i - (k - 3)2] & \text{if } i \text{ is even}
\end{cases}
\]

We will prove that the induced edge labels obtained by the sum of the labels of end vertices are the first \( n - 1 \) polygonal numbers.

Case (i) Suppose \( i \) is odd.
For \( i = 1, 2, \ldots, n-1 \)

\[ f(u_i) + f(u_{i+1}) = \frac{1}{4} (i - 1) [(k - 2)i - (k - 4)] + \frac{1}{4} (i + 1) [(k - 2)(i + 1) - (k - 3)2] \]
\[ = \frac{1}{2} [(k - 2)i - (k - 4)] \]
\[ = P_k(i). \]

Case(i) Suppose \( i \) is even.
For \( i = 1, 2, 3, \ldots, n-1 \)
\[ f(u_i) + f(u_{i+1}) = \frac{1}{4} (i) [(k - 2)i - (k - 3)2] + \frac{1}{4} (i) [(k - 2)(i + 1) - (k - 4)] \]
\[ = \frac{1}{2} [(k - 2)i - (k - 4)] \]
\[ = P_k(i). \]
Thus the induced edge labels are the first \( n-1 \) polygonal numbers.
Hence \( P_n \) admits polygonal sum labeling.

**Theorem 2.2** : The comb \( P_n \odot K_1 \) admits polygonal sum labeling.

**Proof** :
Let \( P_n : u_1u_2 \ldots u_n \) be a path and \( v_i = u_{i+1}u_i \) \((1 \leq i \leq n-1)\) be the edges. Let \( w_1, w_2, \ldots, w_n \) be the pendant vertices adjacent to \( u_1, u_2, \ldots, u_n \) respectively and \( t_i = u_iw_i \) \((1 \leq i \leq n-1)\) be the edges. For \( i = 1, 2, 3, \ldots, n \), define
\[ f(u_i) = \begin{cases} 
\frac{1}{4} (i - 1) [(k - 2)i - (k - 4)] & \text{if } i \text{ is odd} \\
\frac{1}{4} (i) [(k - 2)i - (k - 3)2] & \text{if } i \text{ is even}
\end{cases} \]
\[ f(w_i) = \frac{1}{2} [(k - 2)(n+i-1)^2 - (k - 4)(n+i-1)] - f(u_i), \quad 1 \leq i \leq n. \]
We see that the induced edge labels are the first \( 2n-1 \) polygonal numbers.
Hence \( P_n \odot K_1 \) admits polygonal sum labeling.

**Theorem 2.3** : The star graph \( K_{1,n} \) admits polygonal sum labeling.

**Proof** :
Let \( v \) be the apex vertex and \( v_1, v_2, \ldots, v_n \) be the pendent vertex of the star \( K_{1,n} \).
Define \( f \) by
\[ f(v) = 0, \]
and
\[ f(v_i) = \frac{1}{2} [(k - 2)i - (k - 4)], \quad 1 \leq i \leq n. \]
We see that the induced edge labels are the first \( n \) polygonal numbers.
Hence \( K_{1,n} \) admits polygonal sum labeling.

**Theorem 2.4** : \( S(K_{1,n}) \), the subdivision of the star \( K_{1,n} \) admits polygonal sum labeling.

**Proof** :
Let \( V(S(K_{1,n})) = \{ v, v_i, u_i : 1 \leq i \leq n \} \) and \( E(S(K_{1,n})) = \{ vv_i, v_iu_i : 1 \leq i \leq n \} \)
Define \( f \) by
f(v) = 0.
f(v_i) = \frac{1}{2} \left( (k-2)i^2 - (k-4)i \right), \quad 1 \leq i \leq n.
f(u_i) = \frac{1}{2} \left( (k-2)in^2 + 2(k-2)in - (k-4)n \right), \quad 1 \leq i \leq n.

We will prove that the induced edge labels obtained by the sum of the labels of end vertices are the first 2n polygonal numbers.

\begin{align*}
f(v) + f(v_i) &= 0 + \frac{1}{2} \left( (k-2)i^2 - (k-4)i \right) \\
&= \frac{1}{2} \left( (k-2)i^2 - (k-4)i \right) \\
&= P_k(i). \\
f(v) + f(v_i) &= \frac{1}{2} \left( (k-2)i^2 - (k-4)i \right) + \frac{1}{2} \left( (k-2)n^2 + 2(k-2)in - (k-4)n \right) \\
&= \frac{1}{2} \left( (k-2)(i+n)^2 - (k-4)(i+n) \right) \\
&= P_k(i+n).
\end{align*}

Thus the induced edge labels are the first 2n centered polygonal numbers. Hence $S(K_{1,n})$ admits centered polygonal sum labeling.

**Theorem 2.5** : The bistar $B_{m,n}$ admits polygonal sum labeling.

**Proof** :

Let $V(B_{m,n}) = \{ u,v,u_i,v_j : 1 \leq i \leq m, 1 \leq j \leq n \}$ and $E(B_{m,n}) = \{ uv, uu_i vv_j : 1 \leq i \leq m, 1 \leq j \leq n \}$.

Define $f$ by

\begin{align*}
f(u) &= 0, \\
f(v) &= 1, \\
f(v_i) &= \frac{1}{2} \left( (k-2)(i+1)^2 - (k-4)(i+1) \right), \quad 1 \leq i \leq m, \\
f(u_i) &= \frac{1}{2} \left( (k-2)(m+j+1)^2 - (k-4)(m+j+1) \right) - 1, \quad 1 \leq j \leq n.
\end{align*}

We see that the induced edge labels are the first $m+n+1$ polygonal numbers. Hence $B_{m,n}$ admits polygonal sum labeling.

**Theorem 2.6** : The graph $S_{m,n,r}$ admits centered polygonal sum labeling.

**Proof** :

Let $u_1 u_2 \ldots u_{r+1}$ be a path of length $r$ ($r \geq 1$). Let $v_1, v_2, \ldots, v_m$ be the vertices adjacent to $u_1$ and $w_1, w_2, \ldots, w_n$ be the vertices adjacent to $u_{r+1}$.

For $i = 1, 2, 3, \ldots, r$, define

\begin{align*}
f(u_i) &= \begin{cases} 
\frac{1}{4} (i-1) [(k-2)i - (k-4) ] & \text{if } i \text{ is odd} \\
\frac{1}{4} (i-1) [(k-2)i - (k-3)2] & \text{if } i \text{ is even}
\end{cases} \\
f(v_j) &= \frac{1}{2} \left( (k-2)(r+j)^2 - (k-4)(r+j) \right), \quad 1 \leq j \leq m, \\
f(w_l) &= \frac{1}{2} \left( (k-2)(r+m+l)^2 - (k-4)(r+m+l) \right) - f(u_i), \quad 1 \leq l \leq n.
\end{align*}
We see that the induced edge labels are the first \(m+n+l\) polygonal numbers. Hence \(S_{m,n,r}\) admits polygonal sum labeling.

**Theorem 2.7**: Coconut tree admits centered polygonal sum labeling.

**Proof**:  
Let \(u_1u_2 \ldots u_{i+1}\) be a path of length \(i\) (\(i \geq 1\)). Let \(u_{i+1},u_{i+2}, \ldots ,u_n\) be the pendent vertices, being adjacent to \(u_{i+1}\).  

For \(0 \leq j \leq i\), define  
\[
\begin{align*}
  f(u_j) &= \begin{cases} 
  \frac{1}{4}(j-1)((k-2)j-(k-4)) & \text{if } j \text{ is odd} \\
  \frac{1}{4}(j)((k-2)j-(k-3)2) & \text{if } j \text{ is even}
  \end{cases}
\end{align*}
\]
and for \(i+2 \leq m \leq n\), define  
\[
  f(u_m) = \frac{1}{2} \left( (k-2)m^2 - (k-4)m \right).
\]

Case (i) Suppose \(j\) is odd.  
For \(j = 1, 2, \ldots , i\)  
\[
  f(u_j) + f(u_{j+1}) = \frac{1}{4}(j-1)((k-2)j-(k-4)) + \frac{1}{4}(j+1)((k-2)(j+1)-(k-3)2)
\]
\[
= \frac{1}{2} \left( (k-2)j-(k-4) \right)
= P_k(j).
\]

Case (i) Suppose \(j\) is even.  
For \(j = 1, 2, 3, \ldots , i\)  
\[
  f(u_j) + f(u_{j+1}) = \frac{1}{4}(j)((k-2)j-(k-3)2) + \frac{1}{4}(j)((k-2)(j+1)-(k-4)2)
\]
\[
= \frac{1}{2} \left( (k-2)j-(k-4) \right)
= P_k(j).
\]

and for \(j=1, m = i+1, i+2, \ldots , n\)  
\[
  f(u_j) + f(u_m) = 0 + \frac{1}{2} \left( (k-2)m^2 - (k-4)m \right)
\]
\[
= \frac{1}{2} \left( (k-2)m^2 - (k-4)m \right)
= P_k(m).
\]

We see that the induced edge labels are the first \(n\) polygonal numbers. Hence Coconut tree admits polygonal sum labeling.

**REFERENCES**

2. Frank Harary, Graph theory, Narosa publishing House (2001).