**Journal of Computer and Mathematical Sciences,** *Vol. 6(1), 25-32, January 2015* *ISSN 0976-5727 (Print)*
*www.compmath-journal.org* *ISSN 2319-8133 (Online)*

# On The Tree and Star Numbers of A Graph

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(Received on: January 1, 2015)

## ABSTRACT

The tree number $t(G)$ of a graph $G$ is the minimum number of subsets into which the edge set of $G$ can be partitioned so that each subset induces a tree. The star number $s(G)$ of a graph $G$ is the minimum number of subsets into which the edge set of $G$ can be partitioned so that each subset induces a star. In this paper, the exact values of $t(G)$ and $s(G)$ for some standard graphs are obtained. Bounds on these parameters are established. Also Nordhaus- Gaddum type results for the star number are established.

**Mathematics subject Classification:** 05*C*.

**Keywords:** tree number, star number, path number.

## 1. INTRODUCTION

The graphs considered here are finite, undirected without loops and multiple edges. For any undefined term or notation, we refer Harary[1] and Kulli[2].

The tree number $t(G)$ of a graph $G$ is the minimum number of subsets into which the set of $G$ can be partitioned so that each subset induces a tree. The path number $p(G)$ of a graph $G$ is the minimum number of subsets into which the edge set of $G$ can be partitioned so that each subset induces a path.

The star number $s(G)$ of a graph $G$ is the minimum number of subsets into which the edge set of $G$ can be partitioned so that each subset induces a star. This concept was introduced by Kulli and Sigarkanti in[7].

## 2. TREE NUMBER

**Proposition 1.** For a forest $G$ with $n$ components and without isolated vertices, $t(G) = n$.

**Proposition 2.** For an unicycle graph $G$, $t(G) = 2$.

As an immediate consequence of Proposition 2, we have the following result.

**Corollary 3.** If $G$ is a graph with at least one cycle, then $2 \leq t(G)$.

**Proposition 4.** For a wheel $W_p$ with $p \leq 4$ vertices, $t(W_p) = 2$.

**Proof:** By Corollary 3, we have $2 \leq t(W_p)$. (1)

Let $v_1, v_2, ...., v_p$ be the vertices of $W_p$ such that deg $v_p = p - 1$ and deg $v_i = 3$, $i = 1, 2, ...., p - 1$.
Let $c_i = v_i v_{i+1}$, $i = 1, 2, ..., p - 2$ and $c_{p-1} = v_{p-1} v_1$ and $e_i = v_p v_i$, $i = 1, 2, ..., p - 1$.
Let $E_1 = \{e_1, e_2, ..., e_{p-2}, c_{p-2}\}$
and $E_2 = \{c_1, c_2, ..., c_{p-3}, c_{p-1}, e_{p-1}\}$.

The induced subgraphs $\langle E_1 \rangle$ and $\langle E_2 \rangle$ are trees. Thus by (1), it follows that $t(W_p) = 2$.

The following result is established in[10].

**Proposition A[10]**. For a complete graph $K_p$ with $p$ vertices, $p(K_p) = \left\lceil \frac{p}{2} \right\rceil$.

**Proposition 5.** For a path $P_p$ with $p$ vertices, $t(\overline{P_p}) = \left\lceil \frac{n}{2} \right\rceil - 1$.

where $\overline{P_p}$ is the complement of $P_p$.

**Poof:** Let $F = \{F_1, F_2, ..., F_m\}$ be a path partition of $K_p$. Then by Proposition A, $|E| = p(K_p) = \left\lceil \frac{p}{2} \right\rceil$. Let $P_p = F_i$ for some $i$. Then $F' = \{F_1, F_2, ..., F_{i+1}, ..., F_m\}$ is a minimum tree partition of $\overline{P}_p$.

Thus $t(\overline{P}_p) = |F'| = |F| - 1 = \left\lceil \frac{p}{2} \right\rceil - 1$.

We now establish an upper bound on $t(G)$.

**Theorem 6.** For a $(p, q)$ graph $G$, $t(G) \leq q - p + 2$.
and this bound is sharp.

**Proof:** Let $G$ be a $(p, q)$ graph. Let T be a spanning tree of G. Let $e_1, e_2, \ldots, e_{q-(p-1)}$ be the edges of $G$ not belonging in $E(T)$. Thus $F = \{e_1, e_2, \ldots, e_{q-p+1}, E(T)\}$ is a tree partition of $G$. Hence $t(G) \leq |F| = q - p + 2$.

This bound is sharp as can be seen with trees and unicyclic graphs achieving this bound.

The following result involving the chromatic number $\chi(G)$ gives a lower bound for $t(G)$.

**Theorem 7.** For any graph $G$, $\frac{\chi(G)}{2} \leq t(G)$.

**Proof:** Let F = $\{T_1, T_2, \ldots, T_n\}$ be a minimum tree partition of $G$. Then $t(G) = |F| = n$.
Since the chromatic number of each $T_i$ $(1 \leq i \leq n)$ is two, so that

$$\chi(G) = \sum_{i=1}^{n} \chi(T_i) = \sum_{i=1}^{n} 2 = 2n. \tag{2}$$

From (1) and (2), we have $\frac{\chi(G)}{2} \leq t(G)$.

In the next result, we show that the tree number can be increased by removal of edges.

**Theorem 8.** If $e$ is a bridge of $G$ and $G - e$ contains no isolated vertices, then
$t(G - e) = t(G) + 1$.

**Proof:** Let $F=\{T_1, T_2, \ldots, T_n\}$ be a tree partition of $G$. Then $t(G) = n$.
Let $e \in T_i$ be a bridge of G. Then $e$ is not an end edge of the induced subgraph $\langle T_i \rangle$. Let $e' \in T_{i+1}$ be an edge adjacent to $e$. Then
F'$=\{T_1, T_2, \ldots, T_i \cup T_{i+1}, T_{i+2}, \ldots, T_n\}$
is a tree partition of $G$, which is a contradiction. Thus

$$F'' = \{T_1, T_2, \ldots, T_{i_1}, T_{i_2}, T_{i+1}, \ldots, T_n\}$$

is a minimum tree partition of $G - e$, where $T_{i_1} \cup T_{i_2} \cup \{e\} = T_i$. Therefore

$t(G-e) = |F''|$
$= n + 1$
$= t(G) + 1.$

**3. STAR NUMBER**

We obtain the exact values of star numbers for some standard graphs.

**Proposition 9.** For a nontrivial path $P_p$ with $p$ vertices,

$$s(P_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

**Proof:** Clearly each pair of adjacent edges in $P_p$ induces a star. We consider two cases.

**Case 1.** Suppose $p$ is odd. Then there are $\frac{p-1}{2}$ pairs of adjacent edges in $P_p$. Then

$$s(P_p) = \frac{p-1}{2} = \left\lfloor \frac{p}{2} \right\rfloor.$$

**Case 2.** Suppose $p$ is even. Then there are $\frac{p-2}{2}$ pairs of adjacent edges and a single edge also induces a star. Then

$$s(P_p) = \frac{p-2}{2} + 1 = \frac{p}{2}.$$

Thus in each case,

$$s(P_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

**Proposition 10.** For a cycle $C_p$ with $p \geq 3$ vertices,

$$s(C_p) = \left\lceil \frac{p}{2} \right\rceil.$$

**Proposition 11.** For a complete bipartite graph $K_{m,n}$ $1 \leq m \leq n$, $s(K_{m,n}) = m$.

**Proof :** Le $V=V_1 \cup V_2$ be the vertex set of $K_{m,n}$ such that $|V_1|=m$, $|V_2|=n$, $1 \leq m \leq n$. The edges incident to each vertex of $V_1$ induces a star. Thus since $m \leq n$, $S(K_{m,n}) = m$.

**Proposition 12.** For a complete graph $K_p$ with $p \geq 2$ vertices, $s(K_p) = p-1$.

**Proof:** The result can be verified for $p\geq 3$.

Assume the result is true for all complete graphs with $n > 3$ vertices. Let $V_n=\{v_1, v_2, \ldots, v_n\}$ and $V_{n+1}=\{v_1, v_2, \ldots, v_{n+1}\}$ be the vertex sets of $K_n$ and $K_{n+1}$ respectively. Let $S=\{S_1, S_2, \ldots, S_{n-1}\}$ be the minimum star partition of $K_n$. Let $S_n$ be the set of edges incident to $v_{n+1}$. Then $S_n= K_{1,n}$ is a star. Thus $S'=\{S_1, S_2, \ldots, S_{n-1}, S_n\}$ is the minimum star partition of $K_{n+1}$. Thus.

$s(K_{n+1})=| S | = n$.

Hence the result is true for $p=n+1$.

Thus for $K_p$ with $p$ vertices, $s(K_p)= p - 1$.

**Theorem 13.** For any subgraph H of a graph $G$, $s(H) \leq s(G)$.

**Proof:** Let $S=\{S_1, S_2, \ldots, S_n\}$ be a minimum star partition of $G$. Then $\{S'_1, S'_2, \ldots, S'_m\}$, where $S'_i \subseteq S'_i$, $1\leq m\leq n$, is a minimum star partition of H. Thus $|S'| \leq |S|$. Hence $s(H) \leq s(G)$.

We now establish a lower bound on $s(G)$.

**Theorem 14.** For any graph G, $\frac{diam(G)}{2} \leq s(G)$.

**Proof:** Let $P$ be a path with diam $(G) + 1$ vertices. By Proposition 9 and Theorem 13, it follows that

$$\frac{diam(G)}{2} \leq s(G).$$

We establish another lower bound for regular graphs on $s(G)$.

**Theorem 15.** If $G$ is an r-regular graph with $p\geq 2$ vertices, then $\frac{p}{2} \leq s(G)$.

**Proof:** Let $s(G)=n$. Each subset in a star partition of $G$ induces a star with at most $r$ edges. Therefore

$$nr \geq q. \quad (1)$$

Since $G$ is $r$-regular, it follows that

$$q = \frac{pr}{2}. \quad (2)$$

From (1) and (2), we get $n \geq \frac{p}{2}$.

Hence $s(G) \geq \frac{p}{2}$.

Let $w(G)$ be the clique number of $G$.
We obtain a lower bound for $s(G)$.

**Theorem 16.** For any graph $G$, $w(G) - 1 \leq s(G)$.

**Proof:** Let $H$ be a maximum order of a complete subgraph of $G$. Then by Proposition 12, we have
$s(\mathrm{H}) = w(G) - 1$.
By Theorem 13, we have
$s(H) \leq s(G)$.
Therefore $\quad w(G) - 1 \leq s(G)$.

The independence number $\beta_0(G)$ of $G$ is the maximum cardinality of an independent set of vertices. The vertex covering number $\alpha_0(G)$ of $G$ is the minimum number of vertices in a vertex cover.

Now we obtain some upper bounds for $s(G)$.

**Theorem 17.** For any graph $G$,
i) $s(G) \leq p - \beta_0(G)$.
ii) $s(G) \leq \alpha_0(G)$.

**Proof:** Let $D$ be an independent set in $G$ such that $|D| = \beta_0(G)$.
Suppose for each vertex $v$ in $V–D$, there is a star with $v$ as a cut vertex. Then there exist at most $|V - D|$ stars. Hence
$s(G) \leq |V - D|$.

Thus $\quad s(G) \leq p - \beta_0(G)$.
Since $\alpha_0(G)+ \beta_0(G)=p$, it follows that
$s(G) \leq \alpha_0(G)$.

Nordhaus-Gaddum type results were obtained for several parameters for example[3, 4, 5, 6, 8, 9].

We now establish Nordhaus- Gaddum type results.

**Theorem 18.** For any graph $G$,

$$w(G)+\beta_0(G)-2 \leq s(G)+s(\overline{G}) \leq 2p-(\beta_0(G)+w(G))$$

and

$$\left(w(G)-1\right)\left(\beta_0(G)-1\right) \leq s(G)\,s\left(\overline{G}\right)$$
$$\leq \left(p-\beta_0\left(\overline{G}\right)\right)\left(p-w(G)\right).$$

**Proof:** By Theorem 16, we have

$w(G) - 1 \leq s(G)$. (1)

Let $\overline{G}$ be the complement of $G$. From (1), we have

$$w\left(\overline{G}\right)-1 \leq s\left(\overline{G}\right).$$

Since $w\left(\overline{G}\right)=\beta_0(G)$. it follows that

$w\left(\overline{G}\right)-1 \leq s\left(\overline{G}\right)$. (2)

By Theorem 17. we have

$s(G) \leq p - \beta_0(G)$. (3)

Then $s\left(\overline{G}\right) \leq p-\beta_0\left(\overline{G}\right)$.

Thus $s\left(\overline{G}\right) \leq p-w(G)$. (4)

Form (1), (2), (3) and (4), the result follows.

**Theorem 19.** For connected graphs $G$ and $\overline{G}$,

$s(G)+s(\overline{G}) \leq p(p-3)$.

Furthermore, the equality holds if only if $G=P_4$.

**Proof:** By Theorem 17(ii), $s(G) \leq \alpha_0(G)$. Since both $G$ and $\overline{G}$ are connected, $\Delta(G)$, $\Delta(\overline{G}) < p-1$. This implies that $\beta_0(G)$, $\beta_0(\overline{G}) \geq 2$. Thus $s(G) \leq p - 2 = 2(p-1) - p = 2q - p$. Similarly $s(\overline{G}) \leq 2\overline{q} - p$. Hence $s(G)+s(\overline{G}) \leq 2\left(q+\overline{q}\right)-2p = p(p-1)-2p = p(p-3)$.

Suppose the equality holds. Thus it follows that $s(G) = 2q - p$ and $s(\overline{G}) = 2\overline{q} - p$. This implies that $q$, $\overline{q} < p$. Thus both $G$ and $\overline{G}$ are trees. Thus $G=P_4$.

## REFERENCES


1. F. Harary, *Graph Theory*, Addison Wesley Reading Mass. (1969).

2. V. R. Kulli, *Theory of Domination in Graphs*, Vishwa International Publications, Gulbarga. India (2010).
3. V. R. Kulli, On nonbondage numbers of a graph, *Inter. J. Advanced Research in Computer Sci. and Technology*, 1(1), 42-45 (2013).
4. V. R. Kulli, Edge entire domination in graphs, *Inter. J. Mathematical Archive* 5(10), 275-278 (2014).
5. V. R. Kulli and B. Janakiram, The cobondage number of a graph, *Discuss. Math.* 16, 111-117 (1996).
6. V. R. Kulli, B. Janakiram and R. B. Iyer, Regular number of a graph, *J. Discrete Mathematical Sciences and Cryptography*, 4(1), 57-64 (2001).
7. V. R. Kulli and S. C. Sigarkanti, Tree and star numbers of a graph, Technical Report, 1992:02, Dept. Math. Gulbarga University, Gulbarga, India (1992).
8. V. R. Kulli, S. C. Sigarkanti and N. D. Soner, Entire domination in graphs. In *Advances in Graph Theory*, V. R. Kulli, ed., Vishwa International Publications, Gulbarga, India, 237-243 (1991).
9. V. R. Kulli and N. D. Soner, Complementary edge domination in graphs, *Indian J. Pure Appl. Math.* 28, 917-920 (1997).
10. R. G. Stanton, L. O. James and D. D. Cowan, Same results on path numbers, Proc. Louisiana Conf. *Combinatorics, Graph Theory and Computing*, 112-135 (1970).