Secure and Inverse Secure Total Restrained Domination in Graphs

V. R. Kulli

Department of Mathematics,
Gulbarga University, Gulbarga 585106, INDIA.
email: vrkulli@gmail.com

(Received on: August 22, 2016)

ABSTRACT

A secure total restrained dominating set of a graph $G = (V, E)$ is a total restrained dominating set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $v \in D$ adjacent to $u$ such that $(D - \{v\}) \cup \{u\}$ is a total restrained dominating set. The secure total restrained domination number $\gamma_{str}(G)$ of $G$ is the minimum cardinality of a secure total restrained dominating set. Let $D$ be a minimum secure total restrained dominating set of $G$. If $V - D$ contains a secure total restrained dominating set $D'$ of $G$, then $D'$ is called an inverse secure total restrained dominating set with respect to $D$. The inverse secure total restrained domination number $\gamma_{str}^{-1}(G)$ of $G$ is the minimum cardinality of an inverse secure total restrained dominating set of $G$. In this paper, we initiate a study of these two parameters. We establish a Nordhaus-Gaddum type result.

Mathematics Subject Classification: 05C69, 05C78.

Keywords: Secure dominating set, secure total restrained domination set, secure total restrained domination number, inverse secure total restrained domination number.

1. INTRODUCTION

By a graph $G=(V, E)$, we mean a finite, undirected without isolated vertices, loops and multiple edges. We follow the notation and terminology of $^1$.

A set $D$ of vertices in $G$ is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Recently several domination parameters are given in the books by Kulli in$^{2,3,4}$.

A secure dominating set of $G$ is a dominating set $D \subseteq V$ with the property that for each $u$ in $V - D$, there exists $v$ in $D$ adjacent to $u$ such that $(D - \{v\}) \cup \{u\}$ is a dominating set. The
secure domination number $\gamma_s(G)$ of $G$ is the minimum cardinality of a secure dominating set of $G$. This concept was studied and introduced by Cockayne \textit{et al.} in\textsuperscript{5}. Many other secure domination parameters were studied, for example, in\textsuperscript{6,7,8}.

A set $D \subseteq V$ is a total dominating set if every vertex in $V$ is adjacent to some vertex in $D$. The total domination number $\gamma(G)$ of $G$ is the minimum cardinality of a total dominating set of $G$. A set $D \subseteq V$ is a total restrained dominating set if every vertex of $V$ is adjacent to a vertex in $D$ and every vertex in $V - D$ is also adjacent to a vertex in $V - D$. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of $G$. This concept was introduced by Telle and Proskurowski in\textsuperscript{9} and was studied in\textsuperscript{10,11}.

Let $D$ be a minimum dominating set of $G$. If $V - D$ contains a dominating set $D'$ of $G$, then $D'$ is called an inverse dominating set of $G$ with respect to $D$. The inverse domination number $\gamma^{-1}(G)$ of $G$ is the minimum cardinality of a dominating set in $G$. This concept was introduced by Kulli and Sigarkanti in\textsuperscript{12}. Many other inverse domination parameters were studied, for example, in\textsuperscript{13, 14, 15, 16, 17, 18, 19, 20}.

The join of two graphs $G$ and $H$ is denoted by $G + H$ and it consists of $G \cup H$ and all edges joining every vertex of $G$ with every vertex of $H$.

In this paper, we define the secure total restrained domination number and the inverse secure total restrained domination number of a graph.

### 2. SECURE TOTAL RESTRAINED DOMINATION

We introduce the secure total restrained domination of a graph.

**Definition 1.** A secure total restrained dominating set of a graph $G$ is a total restrained dominating set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $v \in D$ adjacent to $u$ such that $(D - \{v\}) \cup \{u\}$ is a total restrained dominating set. The secure total restrained domination number $\gamma_{tr}(G)$ of $G$ is the minimum cardinality of a secure total restrained dominating set.

Note that $\gamma_{tr}(G)$ is defined only if $G$ has no isolated vertices. Henceforth we consider only graphs with no isolated vertices.

**Definition 2.** The upper inverse secure total domination number $\Gamma_{tr^{-1}}(G)$ of $G$ is the maximum cardinality of a secure total restrained dominating set of $G$.

A $\gamma_{tr^{-1}}$-set is a minimum secure total restrained dominating set.

**Proposition 3.** Let $G$ be a connected graph with $p \geq 2$ vertices. Then $\gamma_n(G) \leq \gamma_{tr}(G)$.

**Proposition 4.** If $D$ is a secure total restrained dominating set of a graph $G$, then $D$ is a secure restrained dominating set of $G$. 

\[ \text{Definition 3.} \text{ The result follows.} \]

\[ \text{Proposition 5.} \text{ If } K_p \text{ is a complete graph with } p \geq 4 \text{ vertices, then} \]
\[ \gamma_{str}(K_p) = 2. \]

\[ \text{Proof:} \text{ Let } D = \{u_1, u_2\} \subseteq V(K_p). \text{ Then } D \text{ is a secure total restrained dominating set of } K_p. \text{ Since} \]
\[ \gamma_{str}(K_p) \geq 2, \text{ the result follows.} \]

\[ \text{Proposition 6.} \text{ If } K_{m,n} \text{ be a complete bipartite graph with } 4 \leq m \leq n, \text{ then} \]
\[ \gamma_{str}(K_{m,n}) = 4. \]

\[ \text{Proof:} \text{ Let } V_1 \cup V_2 \text{ be the vertex set of } K_{m,n} \text{ where } V_1 = \{u_1, u_2, \ldots, u_m\} \text{ and } V_2 = \{v_1, v_2, \ldots, v_n\}. \text{ Let} \]
\[ D = \{u_1, u_2, v_1, v_2\} \subseteq V_1 \cup V_2. \text{ Then } D \text{ is a secure total restrained dominating set of } K_{m,n}. \text{ Then} \]
\[ \gamma_{str}(K_{m,n}) \leq 4. \text{ Suppose } \gamma_{str}(K_{m,n}) = 3. \text{ Without loss of generality, } D_1 = \{u_1, u_2, v_1\}. \text{ Then} \]
\[ D_1 \text{ is not a secure total restrained dominating set, so that } \gamma_{str}(K_{m,n}) = 4. \text{ Hence the result follows.} \]

\[ \text{Proposition 7.} \text{ If } W_p \text{ is a wheel with } n \geq 4 \text{ vertices, then} \]
\[ \gamma_{str}(W_p) = p - 2. \]

\[ \text{Proof:} \text{ Let } V(W_p) = \{u_1, u_2, \ldots, u_p\}, \deg u_1 = p - 1 \text{ and } \deg u_i = 3, \text{ for } 2 \leq i \leq p. \text{ Let} \]
\[ D = \{u_1, u_2, \ldots, u_{p-2}\} \subseteq V(W_p). \text{ Since } D \text{ is a secure total restrained dominating set of } W_p, \text{ it follows that} \]
\[ \gamma_{str}(W_p) \leq p - 2. \text{ Suppose } \gamma_{str}(W_p) = p - 3. \text{ Then without loss of generality, } D_1 = \{u_1, u_2, \ldots, u_{p-3}\}. \text{ Clearly } D_1 \text{ is not a secure total restrained dominating set, so that } \gamma_{str}(W_p) = p - 2. \text{ Hence} \]
\[ \text{the result follows.} \]

\[ \text{3. INVERSE SECURE TOTAL RESTRAINED DOMINATION} \]

\[ \text{We introduce the concept of inverse secure total restrained domination as follows:} \]

\[ \text{Definition 8.} \text{ Let } G=(V, E) \text{ be a graph. Let } D \text{ be a minimum secure total restrained dominating set of } G. \text{ If} \]
\[ V - D \text{ contains a secure total restrained dominating set } D' \text{ of } G, \text{ then } D' \text{ is called an} \]
\[ \text{inverse secure total restrained dominating set with respect to } D. \text{ The inverse secure total} \]
\[ \text{restrained domination number of } G, \text{ denoted by } \gamma_{str}^{-1}(G), \text{ is the smallest cardinality of an} \]
\[ \text{inverse secure total restrained dominating set of } G. \]

\[ \text{Definition 9.} \text{ The upper inverse secure total restrained domination number of } G, \text{ denoted by} \]
\[ \Gamma_{str}^{-1}(G), \text{ is the maximum cardinality of an inverse secure total restrained dominating set of } G. \]

\[ \text{A } \gamma_{str}^{-1}-\text{set is a minimum inverse secure total restrained dominating set.} \]
Example 10. Let $K_4$ be the complete graph. Then $\gamma_{str}(K_4) = 2$ and $\gamma_{str}^{-1}(K_4) = 2$.

Remark 11. Not all graphs have an inverse secure total restrained dominating set.

Theorem 12. Let $D$ be a $\gamma_{str}$-set of a nontrivial connected graph $G$. If a $\gamma_{str}^{-1}$-set exists, then $G$ has at least 4 vertices.

Proof: Let $D$ be a $\gamma_{str}^{-1}$-set of a connected graph $G$. Since $G$ has no isolated vertices, $\gamma_{str}(G) = |D| \geq 2$. If a $\gamma_{str}^{-1}$-set exists, then $V - D$ contains a secure total restrained dominating set with respect to $D$. Thus $|V - D| \geq 2$. Thus $G$ has at least 4 vertices.

We obtain the exact values of $\gamma_{str}^{-1}(G)$ for some standard graphs.

Theorem 13. If $K_p$ is a complete graph with $p \geq 4$ vertices, then $\gamma_{str}^{-1}(K_p) = 2$.

Proof: Let $D$ be a $\gamma_{str}$-set of $K_p$. By Proposition 5, $|D| = 2$. Let $D = \{u, v\}$. Then $S = \{x, y\}$ is a $\gamma_{str}^{-1}$-set of $K_p$ for $x, y \in V(K_p) - \{u, v\}$. Thus $\gamma_{str}^{-1}(K_p) = 2$.

Theorem 14. If $K_{m,n}$ is a complete graph with $4 \leq m \leq n$, then $\gamma_{str}^{-1}(K_{m,n}) = 4$.

Proof: Let $V(K_{m,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. By Proposition 6, $D = \{u_1, u_2, v_1, v_2\}$ is a $\gamma_{str}$-set of $K_{m,n}$. Then $S = \{u_3, u_4, v_3, v_4\}$ is a $\gamma_{str}^{-1}$-set of $K_{m,n}$ for $u_3, u_4, v_3, v_4 \in V(K_{m,n}) - \{u_1, u_2, v_1, v_2\}$. Thus $\gamma_{str}^{-1}(K_{m,n}) = 4$.

Proposition 15. For any graph $G$ with a $\gamma_{str}^{-1}$-set,

\[ \gamma_{str}(G) \leq \gamma_{str}^{-1}(G) \tag{1} \]

and this bound is sharp.

Proof: Every inverse secure total restrained dominating set of $G$ is a secure total restrained dominating set. Thus (1) holds.

The complete graph $K_4$ and the complete bipartite graph $K_{4,4}$, realize the sharp lower bound.

Proposition 16. If a $\gamma_{str}^{-1}$-set exists in a graph $G$ with $p$ vertices, then $\gamma_{str}(G) + \gamma_{str}^{-1}(G) \leq p$ and this bound is sharp.

Proof: This follows from the definition of $\gamma_{str}^{-1}(G)$.

Both graphs $K_4$ and $K_{4,4}$ achieve this bound.

We establish lower and upper bounds on $\gamma_{str}^{-1}(G)$.
Theorem 17. If a $\gamma_{str}^{-1}$-set exists in a graph $G$ with $p$ vertices, then
$2 \leq \gamma_{str}^{-1}(G) \leq p - 2$
and these bounds are sharp.

Proof: By Proposition 15, $\gamma_{str}(G) \leq \gamma_{str}^{-1}(G)$ and since $2 \leq \gamma_{str}(G)$,
$2 \leq \gamma_{str}^{-1}(G)$.
By Proposition 16, $\gamma_{str}^{-1}(G) \leq p - \gamma_{str}(G)$ and since $2 \leq \gamma_{str}(G)$,
$\gamma_{str}^{-1}(G) \leq p - 2$.
Hence the result follows.

The complete graph $K_4$ realizes the sharp lower and upper bounds.

Now we obtain a Nordhaus-Gaddum type result for secure total restrained domination number.

Theorem 18. Let $G$ be a graph with $p \geq 4$ vertices. If both $\gamma_{str}^{-1}(G)$ and $\gamma_{str}^{-1}(\overline{G})$ exist, then
$4 \leq \gamma_{str}^{-1}(G) + \gamma_{str}^{-1}(\overline{G}) \leq 2(p - 2)$
$4 \leq \gamma_{str}^{-1}(G)\gamma_{str}^{-1}(\overline{G}) \leq (p - 2)^2$

Proof: Since both $\gamma_{str}^{-1}(G)$ and $\gamma_{str}^{-1}(\overline{G})$ exist,
$2 \leq \gamma_{str}^{-1}(G)$ and $2 \leq \gamma_{str}^{-1}(\overline{G})$.
Thus both lower bounds hold.

By Theorem 17, we have
$\gamma_{str}^{-1}(G) \leq p - 2$ and $\gamma_{str}^{-1}(\overline{G}) \leq p - 2$.
Thus both upper bounds hold.

We give some graphs for which $\gamma_{str}(G) = \gamma_{str}^{-1}(G)$.

Proposition 19. For any integer $p \geq 4$,
$\gamma_{str}(K_p) = \gamma_{str}^{-1}(K_p) = 2$.

Proposition 20. For any integers $m, n \geq 4$,
$\gamma_{str}(K_{m,n}) = \gamma_{str}^{-1}(K_{m,n}) = 4$.

Proposition 21. For integers $m, n \geq 4$,
$\gamma_{str}(\overline{K_{m,n}}) = \gamma_{str}^{-1}(\overline{K_{m,n}}) = 4$.

Proof: We have $\overline{K_{m,n}} = K_m \cup K_n$. Therefore
$\gamma_{str}(\overline{K_{m,n}}) = \gamma_{str}(K_m) + \gamma_{str}(K_n) = 2 + 2 = 4$. 

424
\[ \gamma_{str}^{-1}(K_{m,n}) = \gamma_{str}^{-1}(K_m) + \gamma_{str}^{-1}(K_n) = 2 + 2 = 4. \]

Hence the result follows.

**Theorem 22.** Let \( G \) and \( H \) be two nontrivial complete graphs. Then 
\[ \gamma_{str}(G+H) = \gamma_{str}^{-1}(G+H) = 2. \]

**Proof:** Let \( G \) and \( H \) be two nontrivial complete graphs. Then \( G+H \) is a complete graph with at least 4 vertices. Hence by Proposition 19, 
\[ \gamma_{str}(G+H) = \gamma_{str}^{-1}(G+H) = 2. \]

**Proposition 23.** Characterized graphs \( G \) for which 
\[ \gamma_{str}(G) = \gamma_{str}^{-1}(G). \]

**Proposition 24.** Characterized graphs \( G \) for which 
\[ \gamma_{str}(G) + \gamma_{str}^{-1}(G) = p. \]

**REFERENCES**