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Vulnerability: Vertex Neighbor Integrity of Middle Graphs

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ABSTRACT

The vertex neighbor integrity of a connected graph $G = (V,E)$ is denoted as $VNI(G)$ and defined by

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G/S)\},$$

where S is any vertex subversion strategy of G and $m(G/S)$ is the number of vertices in the largest component of G/S. In this paper we obtain vertex neighbor integrity of middle graph of some standard graphs and combinations of these graphs.

AMS Classifications: 05C40, 05C70, 05C76.

Keywords: Vulnerability, Stability, Vertex Neighbor Integrity, Middle Graph.

1. INTRODUCTION

The vulnerability of a communication network measures the resistance of network to the disruption of operation after the failure of certain stations or communication links. For any communication network greater degrees of stability or less vulnerability is required. Vulnerability can be measured by certain parameters like connectivity, toughness, integrity, binding number[2,3,4,7] etc. In the analysis of vulnerability of communication network to disruption following two parameters are of great importance.

The size of the largest remaining group within which mutual communication can still occur. The number of elements that are not functioning, but these parameters do not consider the effect which removal of a vertex has on the neighbors of the vertex. For a simple measure of how easily a graph can be broken apart, one needs to look no further than connectivity. However, neither vertex-connectivity nor edge-connectivity measures how easy it is to break the graph into small pieces. One measure of how easily this can be done is neighbor integrity. Neighbor integrity is a measure of the vulnerability of graphs to disruption caused by the removal of vertices and all of their adjacent vertices. The concept of the vertex neighbor-integrity was introduced as a measure of graph vulnerability by M.B.Cozzens and Shu-Shih Y Wu[5,8,9]. Let u be a vertex of a graph $G = (V, E)$. Then $N(u) = \{v \in V(G), v \text{ and } u \text{ are adjacentg}\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ denotes the closed neighborhood of u. The line graph $L(G)$ of a graph G is a graph having the edge set of G as its vertex set and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common[10]. The middle graph $M(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and edges of G in which two vertices are adjacent if they are adjacent edges of G or one is a vertex of G and other is an edge of G incident with it[1].

A vertex u of a graph G is said to be subverted if the closed neighborhood $N[u]$ is deleted from G. A set of vertices $S = \{u_1, u_2, \ldots, u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. Let G/S be the survival subgraph when S has been a vertex subversion strategy of G[8]. The vertex neighbor integrity of a graph G, $VNI(G)$, is defined to be

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G/S)\},$$

where S is any vertex subversion strategy of G and $m(G/S)$ is the number of vertices in the largest component of G/S. The set S is called the *VNI*-set of a graph G, which gives its neighbor integrity[8]. The vertex neighbor integrity of line graph is discussed by Vecdi Aytac[12]. This article includes some results on the vertex neighbor integrity of middle graphs and its composition graphs.

2. VERTEX NEIGHBOR INTEGRITY OF MIDDLE GRAPHS OF SOME STANDARD GRAPHS

In this section, we obtain neighbor integrity of middle graph of some standard graphs. Then, we calculate neighbor integrity of middle graph of Cartesian product of some graphs.

Theorem 2.1. *Let $M(P_n)$ be the middle graph of a path P_n. Then*

$$VNI(M(P_n)) = \lceil 2\sqrt{2n+4} \rceil - 6$$

for every $n \geq 11$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Proof. The number of vertices in $M(P_n)$ is $2n - 1$. If we remove r vertices from $M(P_n)$, then at least one of the remaining connected components has $\frac{2n-1-5r}{r+1}$ vertices. Since $\frac{2n-1-5r}{r+1} \geq 0$, it follows that r must be at most $\frac{2n-1}{5}$. So, $VNI(M(P_n)) \geq min_r \{r + \frac{2n-1-5r}{r+1}\}$, where $r \leq \frac{2n-1}{5}$.

The function $f(r) = r + \frac{2n-1-5r}{r+1}$ takes its minimum value at $r = -1 + \sqrt{2n+4}$ and since $-1 + \sqrt{2n+4} \leq \frac{2n-1}{5}$, we have $n \geq \frac{21}{2}$.
Consequently, *VNI(M(P$_n$))* = $\lceil 2\sqrt{2n+4} \rceil - 6$ for every $n \geq 11$. Since the neighbor integrity is integer valued, we round up this to get a lower bound and *VNI(M(P$_n$))* = $\lceil 2\sqrt{2n+4} \rceil - 6$ for every $n \geq 11$.

Remark 2.2. The values of *VNI(M(P$_n$))* for $n \leq 10$ are given in the following table:

n	2	3	4	5	6	7	8	9	10
$VNI(M(P_n))$	1	2	2	3	3	3	4	4	4

Theorem 2.3. Let *M(C$_n$)* be the middle graph of *C$_n$*. Then,

$$VNI(M(C_n)) \quad \lceil 2\sqrt{2n} \rceil - 5 \ for\ every\ n \geq 13.$$

Proof. The number of vertices in *M(C$_n$)* is 2*n*. If we remove r vertices from *M(C$_n$)*, then at least one of the remaining connected component has $\frac{2n-5r}{r}$ vertices. Since $\frac{2n-5r}{r} \geq 0$, it follows that *r* must be at most $\frac{2n}{5}$. So, *VNI(M(C$_n$))* $\geq min_r \left\{ r + \frac{2n-5r}{r} \right\}$, where $r \leq \frac{2n}{5}$.
The function $f(r) = r + \frac{2n-5r}{r}$ takes its minimum value at $r = \sqrt{2n}$ and since $\sqrt{2n} \leq \frac{2n}{5}$, we have $n \geq 13$.
Consequently, *VNI(M(C$_n$))* = $2\sqrt{2n} - 5$ for every $n \geq 13$. Since the neighbor integrity is integer valued, we round up this to get a lower bound and *VNI(M(C$_n$))* = $\lceil 2\sqrt{2n} \rceil - 5$ for every $n \geq 13$.
Remark 2.4. The values of *V NI(M(C$_n$))* for $n \leq 12$ are given in the following table:

n	3	4	5	6	7	8	9	10	11	12
$VNI(M(C_n))$	2	2	3	3	4	4	4	5	5	5

Theorem 2.5. Let *M(K$_n$)* be the middle graph of a complete graph $K_n, n \geq 3$. Then,

$$VNI(M(K_n)) = \begin{cases} \frac{n-1}{2} + 1, & if\ n\ is\ odd; \\ \frac{n}{2}, & if\ n\ is\ even. \end{cases}$$

Proof. Let *S* be a vertex subversion strategy of *M(K$_n$)*. We consider the following two cases.

Case 1: n is odd. Since the complete graph K_n has $\frac{n(n-1)}{2}$ edges, it follows that the number of vertices in $M(K_n)$ is $n + \frac{n(n-1)}{2}$. If S has $\frac{n-1}{2}$ number of vertices corresponding to independent edges of K_n, then we get only one vertex.
So, $VNI(M(K_n)) = \frac{n-1}{2} + 1$.
Case 2: n is even. If S has $\frac{n}{2}$ number of vertices corresponding to independent edges of K_n, then the set S covers all vertices in $M(K_n) - S$. So, $VNI(M(K_n)) = \frac{n}{2}$.

Theorem 2.6. Let $M(K_{1,n})$ be the middle graph of a star $K_{1,n}$. Then, $VNI(M(K_{1,n})) = 2$.

Proof. Let S be a vertex subversion strategy of $M(K_{1,n})$. Since $M(K_{1,n})$ contains a complete graph K_n as its subgraph. If we choose the set S as any one vertex of K_n in $M(K_{1,n})$, then there exist $n - 1$ remaining components each contains only one vertex. So, $VNI(M(K_{1,n})) = 2$.

Theorem 2.7. Let $M(W_{1,n+1})$ be the middle graph of the wheel graph $W_{1,n+1}$ of order $n + 2$ vertices. Then, $VNI(M(W_{1,n+1})) = |2\sqrt{2n+4}\,| - 5$ for every $n \geq 11$.

Proof. Since K_{n+2} is an induced subgraph of $M(W_{1,n+1})$. Then if we remove one vertex and its adjacent vertices from K_{n+2} in $M(W_{1,n+1})$ corresponding to edge of $W_{1,n+1}$ we get only one component. This component is $M(P_n)$. So, $VNI(M(W_{1,n+1})) = 1 + VNI(M(P_n))$. Therefore, by Theorem 2.1,

$$VNI(M(W_{1,n+1})) = \lceil 2\sqrt{2n+4} \rceil - 5$$

for every $n \geq 11$.

Remark 2.8. The values of $VNI(M(W_{1,n+1}))$ for $n \leq 10$ are given in the following table:

n	2	3	4	5	6	7	8	9	10
$VNI(M(W_{1,n+1}))$	2	3	3	4	4	4	5	5	5

Theorem 2.9. Let $M(K_2 \times P_n)$ be the middle graph of $K_2 \times P_n$ of order $2n$ vertices. Then, $VNI(K_2 \times P_n)) = n$ for every $n \geq 3$.

Proof. Let D be the set of vertices of maximum degree in $M(K_2 \times P_n)$. Then $|D| = n - 2$. Now, if we remove D vertices from $M(K_2 \times P_n)$, then we get two P_3 graphs. Let S be the vertex subversion strategy of $M(K_2 \times P_n)$. Then $S = D + \{u, v\}$, where u and v are two centers of two P_3. Therefore $VNI(K_2 \times P_n))= |S|= n$ for every $n \geq 3$.

Corollary 2.10. For every $n, n \geq 9$, $VNI(K_2 \times P_n)) \geq 2(VNI(M(P_n)))$ +1.

Theorem 2.11. Let $M(K_2 \times C_n)$ be the middle graph of $K_2 \times C_n$ of order *2n* vertices. Then, $VNI\ M(K_2 \times C_n)) = n$ for every $n \geq 3$.

Proof. Consider the graph $M(K_2 \times C_n)$ as shown in Figure. 2.1. The number of vertices in $M(K_2 \times C_n)$ is *5n*. Let *S* be a vertex subversion strategy of $M(K_2 \times C_n)$. We choose *S* to be the set of n vertices say $v_1, v_2, \ldots\ldots, v_n$ as shown in Figure.2.1. If we remove the set *S* and all its adjacent vertices, we get $VNI\big(M(K_2 \times C_n)\big) = |S| = n$ for every $n \geq 3$.

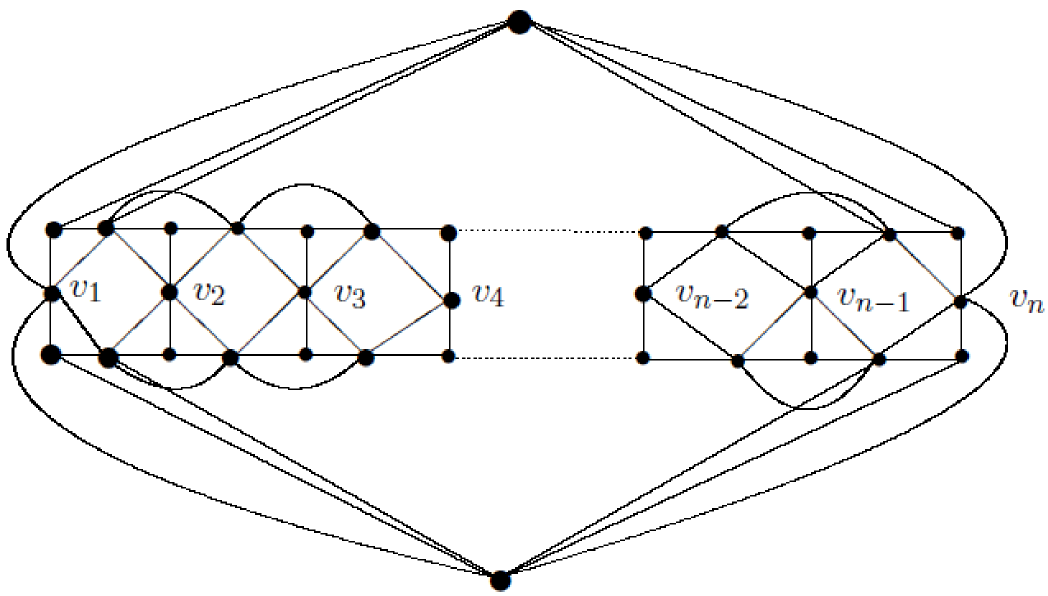

Figure 2.1

Theorem 2.12. Let $M(K_2 \times K_{1,n})$ be the middle graph of $K_2 \times K_{1,n}$ of order *2(n + 1)* vertices. Then, $VNI\left(M\left(K_2 \times K_{1,n}\right)\right) = 4$ for every $n \geq 3$.

Proof. Consider the graph $M(K_2 \times K_{1,n})$ as shown in Figure.2.2. The number of vertices in $M(K_2 \times K_{1,n})$ is $5n + 3$. Let *S* be a vertex subversion strategy of $M(K_2 \times K_{1,n})$. We choose $S = (w\}$ as shown in Figure.2.2. If we remove the set *S* and all its adjacent vertices, then the remaining graph is a components with three vertices. Therefore, $VNI\left(M\left(K_2 \times K_{1,n}\right)\right) = 1 + 3 = 4$ for every $n \geq 3$.

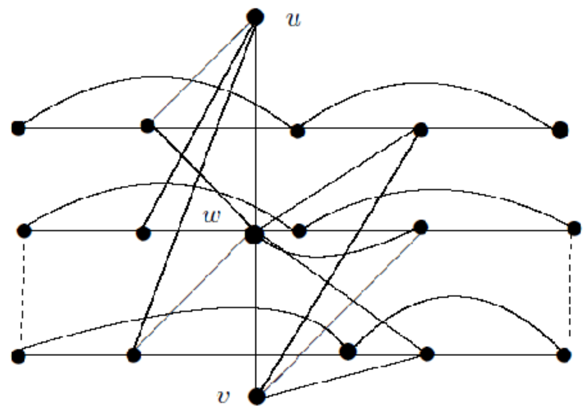

Figure 2.2

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