A Certain Class of p-Valent Analytic Function and Partial Sums

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ABSTRACT

In this paper, we define and study some subclasses of p-valent analytic function of higher order in unit disc. These classes generalize some classes which were previously studied. We obtain coefficient conditions; partial sums $f_n(z)$ of functions $f(z)$ in these classes are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f'_n(z)$ be determined.

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1. INTRODUCTION

Let $A_p$ denotes a class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad p \in \mathbb{C} = \{1,2,3,...\}$$

which are analytic and $p$-valent in the open unit disk $\Delta = \{z \in \mathbb{C}, |z| < 1\}$. Also a subclass of $A_p$ denoted by $T_p$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{p+k}| z^{p+k}$$

Let $g \in A_p$ be of the form:

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \quad b_{p+k} \neq 0$$

Where \( b_{p+k} \leq b_{p+k+1}, k \in \mathbb{N} \)

A function \( f(z) \in A_p \) is said to be p-valently starlike of order \( \alpha \) in \( \Delta \), if it satisfies the inequality
\[
Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha (z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N})
\]

The class of all p-valent starlike functions of order \( \alpha \) is denoted by \( S_p^*(\alpha) \) and we write \( S_1^*(\alpha) \equiv S^*(\alpha) \).

On the other hand, a function \( f(z) \in A_p \) is said to p-valently convex of order \( \alpha \) in \( \Delta \), if it satisfies the inequality
\[
Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha (z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N})
\]

The class of all p-valent convex functions of order \( \alpha \) is denoted by \( K_p(\alpha) \) and write \( K_1(\alpha) \equiv K(\alpha) \).

The classes \( S_p^*(\alpha) \) and \( K_p(\alpha) \) were studied by Patil and Thakre\(^3\) and Owa\(^2\).

Furthermore, a function \( f(z) \in A_p \) is said to be p-valently close-to-convex of order \( \alpha \) in \( \Delta \), if it satisfies the inequality
\[
Re \{ z^{1-p}f'(z) \} > \alpha (z \in \Delta; 0 \leq \alpha < p; p \in \mathbb{N})
\]

The class of all p-valent close-to-convex of order \( \alpha \) is denoted by \( CK_p(\alpha), CK_p(0) \equiv CK_p \) and denote \( CK_1(0) \equiv CK \).

A Hadamard product of two p-valent analytic functions \( f \) & \( g \) of the form (1.1) and (1.3) respectively, is defined by
\[
(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z)
\]  (1.4)

Involving derivatives of the Hadamard product defined in (1.4), the subclass \( K(p, g, \alpha, m) \) of the class \( A_p \) is defined as follows:

For \( 0 \leq \alpha < p, p > m, m \in \mathbb{N}_0 \)

\[
K(p, g, \alpha, m) = \left\{ f(z) \in A_p: \text{Re} \left\{ 1 + \frac{2(f * g)^{m+2}(z)}{(f * g)^{m+1}(z)} + m \right\} > \alpha, z \in \Delta \right\}
\]

where \((f * g)^{r}(z)\) denotes the \( r \)th derivative of \((f * g)(z)\) and is given by
\[
(f * g)^{r}(z) = \frac{p!}{(p-r)!} z^{p-r} + \sum_{k=1}^{\infty} \frac{(p+k)!}{(p+k-r)!} a_{p+k} b_{p+k} z^{p+k-r}, r \in \mathbb{N}_0
\]  (1.5)

We also let \( TK(p, g, \alpha, m) = K(p, g, \alpha, m) \cap T_p \) and for \( m=0 \) and \( g(z) = \frac{z^p}{1-z} \) then the class \( K(p, g, \alpha, m) \) coincides with the classes studied by Srivastava et al. in [5, 6]. We note that taking \( g(z) = \frac{z^p}{1-z}, K(p, g, \alpha, 0) \equiv K_p(\alpha) \).

Following the idea of P. Sharma and P. Srivastava\(^2\) we define the class \( K(p, g, \alpha, m) \) and obtained the coefficient conditions for these classes. Further, results based on partial sums of functions belonging to these classes are derived.
2. MAIN RESULTS

In this section, we obtain coefficient conditions for functions in the classes defined above.

Theorem 1 Let \( g \in A_p \) of the form (1.3), then for \( 0 \leq \alpha < p, p > m, m \in \mathbb{N}_0 \) and \( \zeta \in \mathbb{C}/\{-1\} \) with \( |\zeta| = 1 \), then \( f \in A_p \) of the form (1.1) belongs to the class \( K(p, g, \alpha, m) \) if and only if

\[
1 + \sum_{k=1}^{\infty} A_p z^k \neq 0 \quad (z \in \Delta/\{0\})
\]

Where

\[
A_{p+k} = \frac{(p+k)! (p-m-1)! [k(1+\zeta) + 2(p-\alpha)]}{2(p-\alpha)(p+k-m-1)! p} a_{p+k} b_{p+k}
\]

Proof: Set

\[
p(z) = \frac{1 + z(f*g)^{m+2}(z)}{f*g)^{m+1}(z)} + m - \alpha
\]

which is analytic in \( \Delta \) with \( p(0) = 1 \), then \( f \in K(p, g, \alpha, m) \) if and only if \( \zeta \in \mathbb{C}/\{-1\}, |\zeta| = 1, z \in \Delta \)

\[
1 + \frac{z(f*g)^{m+2}(z)}{f*g)^{m+1}(z)} + m - \alpha \neq \frac{\zeta - 1}{\zeta + 1}
\]

Or,

\[
(1 + \zeta)z(f*g)^{m+2}(z) - [2\alpha + \zeta(p-m) - (p + m + \zeta + 1)](f*g)^{m+1}(z) \neq 0
\]

On using series expansion of \((f*g)^{m+2}\) and \((f*g)^{m+1}\) from (1.5), it is equivalent that

\[
p! 2(p-\alpha) z^{p-m-1} \left\{ \sum_{k=1}^{\infty} \frac{(p+k)! (p-m-1)! [k(1+\zeta) + 2(p-\alpha)]}{2(p-\alpha)(p+k-m-1)! p} a_{p+k} b_{p+k} z^k \neq 0 \right\}
\]

and for \( z \neq 0 \), it is equivalent that

\[
\left\{ \sum_{k=1}^{\infty} \frac{(p+k)! (p-m-1)! [k(1+\zeta) + 2(p-\alpha)]}{2(p-\alpha)(p+k-m-1)! p} a_{p+k} b_{p+k} z^k \neq 0 \right\}
\]

This completes the proof.

Taking \( p=1, g(z) = \frac{z^p}{1-z} \) and \( m = 0 \) in Theorem 1, we get special case as follows.

Corollary 1 A function \( f \in A_p \) of the form (1.1) is in the class \( K(\alpha) \) if and only if

\[
1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0 \quad (z \in \Delta)
\]

Where

\[
A_k = \frac{k^2(\zeta+1)-2\alpha-k(\zeta-1)}{2-2\alpha} a_k, \zeta \in \mathbb{C}/\{-1\}, |\zeta| = 1
\]

Theorem 2 Let \( g \in A_p \) of the form (1.3) and \( 0 \leq \alpha < p, p > m, m \in \mathbb{N}_0 \) a necessary and sufficient condition for a function \( f \in T_p \) of the form (1.2) belongs to \( TK(p, g, \alpha, m) \) is that:

564

\[ \sum_{k=1}^{\infty} \frac{(p+k)!(p+k-%alpha)}{(p-%alpha)!} \phi_k^m p_{p+k} |a_{p+k}| \leq 1, \quad (2.1) \]

Where \[ \phi_k^m = \frac{(p-m-1)!}{(p+k-m-1)!} \]

**Proof.** To prove the sufficient condition for \( f \in TK(p, g, \alpha, m) \), we consider

\[ p(z) = \frac{1 + \frac{z(f*g)^{m+2}(z)}{(f*g)^{m+1}(z)} m - \alpha}{p - \alpha} \]

Now it needs to show that

\[ |p(z) - 1| - |p(z) + 1| \leq 0 \]

We see that

\[ |p(z) - 1| = \frac{|z^{p-m-1}|}{(p-%alpha)\alpha\{(f*g)^{m+1}(z)\}} \left| \sum_{k=1}^{\infty} \frac{(p+k)!a_{p+k}b_{p+k}z^k}{(p+k-m-1)!} \right| \]

\[ \leq \frac{|z^{p-m-1}|}{(p-%alpha)\alpha\{(f*g)^{m+1}(z)\}} \sum_{k=1}^{\infty} \frac{(p+k)!|a_{p+k}|b_{p+k}}{(p+k-m-1)!} \]

and

\[ = \frac{|z^{p-m-1}|}{(p-%alpha)\alpha\{(f*g)^{m+1}(z)\}} \left| \frac{2(p-%alpha)p!}{(p-m-1)!} - \sum_{k=1}^{\infty} \frac{(p+k)!|k+2(p-%alpha)|a_{p+k}b_{p+k}z^k}{(p+k-m-1)!} \right| \]

\[ \geq \frac{|z^{p-m-1}|}{(p-%alpha)\alpha\{(f*g)^{m+1}(z)\}} \left\{ \frac{2(p-%alpha)p!}{(p-m-1)!} - \sum_{k=1}^{\infty} \frac{(p+k)!|k+2(p-%alpha)|a_{p+k}b_{p+k}}{(p+k-m-1)!} \right\} \]

Thus

\[ \leq \frac{|z^{p-m-1}|}{(p-%alpha)\alpha\{(f*g)^{m+1}(z)\}} \left\{ \frac{|p(z) - 1| - |p(z) + 1|}{(p-m-1)!} + \sum_{k=1}^{\infty} \frac{(p+k)!2|k+p-%alpha|a_{p+k}b_{p+k}}{(p+k-m-1)!} \right\} \]

\[ \leq 0, \]

If (2.1) holds. Conversely, to show that condition (2.1) is necessary for \( f(z) \in TK(p, g, \alpha, m) \), let

\[ \text{Re} \left\{ \frac{1 + \frac{z(f*g)^{m+2}(z)}{(f*g)^{m+1}(z)} m - \alpha}{p - \alpha} \right\} > 0 (z \in \Delta) \]

Applying series expansion of \( (f*g)^{m+2} \) and \( (f*g)^{m+1} \) from (1.5), we get

\[ \left\{ \frac{(p-%alpha)p!}{(p-m-1)!} z^{p-m-1} - \sum_{k=1}^{\infty} \frac{(p+k)!|a_{p+k}|b_{p+k}z^{p+k-m-1}}{(p+k-m-1)!} \right\} > 0 (z \in \Delta) \]

565
Or,
\[
Re \left\{ 1 - \sum_{k=1}^{\infty} \frac{(p + k)! (p - m - 1)! (p + k - \alpha)}{(p - \alpha)(p + k - m - 1)! p!} |a_{p+k}| b_{p+k} z^k \right\} > 0 \quad (z \in \Delta),
\]

Letting \( z \to 1^- \) along the real axis, we get
\[
\left\{ 1 - \sum_{k=1}^{\infty} \frac{(p + k)! (p - m - 1)! (p + k - \alpha)}{(p - \alpha)(p + k - m - 1)! p!} |a_{p+k}| b_{p+k} z^k \right\} \geq 0
\]

Which is (2.1). This proves Theorem 2.

**Corollary 2** Let \( g \in A_\alpha \) of the form (1.3), \( 0 \leq \alpha < p \) and \( p > m, m \in \mathbb{N}_0 \) if \( f \in T_p \) of the form (1.2) belongs to the class \( TK(p, g, \alpha, m) \), then
\[
|a_{p+k}| \leq \frac{(p - \alpha)(p + k - m - 1)! p!}{(p + k)! (p - m - 1)! (p + k - \alpha)b_{p+k}} (k, p \in \mathbb{N}, b_{p+k} > 0)
\]
The result is sharp for the function \( f_k(z) \) given by
\[
f_k(z) = \frac{(p - \alpha)(p + k - m - 1)! p!}{(p + k)! (p - m - 1)! (p + k - \alpha)b_{p+k}} z^{p+k} (k, p \in \mathbb{N}, b_{p+k} > 0)
\]

Taking \( g(z) = \frac{z^p}{1 - z} \), \( m=0 \) in Theorem 2, we get special case as follows:

**Corollary 3** A necessary and sufficient conditions for a function \( f \in T_p \) of the form (1.2) to be in the class \( K_p(\alpha) \) \( (0 \leq \alpha < 1) \) is that
\[
\sum_{k=1}^{\infty} \frac{(p+k-\alpha)(p+k)}{(p-\alpha)p} |a_{p+k}| \leq 1.
\]

**Remark.** The result obtained in corollary 3 coincides with the result obtained by P.Srivastava, if we put \( p = 1 \) and \( k = n - 1 \) in the result of corollary 3, then the above result coincides with the result obtained by Silverman and S.Topkaya.

3. SOME CONSEQUENCES OF MAIN RESULTS

A consequence of the theorem 1 follows as a sufficient condition for \( K(p, g, \alpha, m) \)

**Class:**

**Theorem 3** Let \( g \in A_\alpha \) of the form (1.3), then for \( 0 \leq \alpha < p, p > m, m \in \mathbb{N}_0 \) and \( \zeta \in \mathbb{C}/\{0\} \) with \( |\zeta| = 1 \), then \( f \in A_\alpha \) of the form (1.1) satisfies:
\[
1 + \sum_{k=1}^{\infty} B_k z^k \neq 0, \quad (z \in \Delta / \{0\}) \quad (3.1)
\]
Where
\[
B_k = \frac{\beta^k}{k!} + \sum_{j=0}^{k-1} \frac{\beta^j (p + k - j)! (p - m - 1)! [(k - j)(1 + \zeta) + 2(p - \alpha)]}{j! 2(p - \alpha)} a_{p+k-j} b_{p+k-j}
\]

Then \( f(z) \in K(p, g, \alpha, m) \).

**Proof.** Let

\[ A_{p+k-j} = \frac{(p + k - j)! (p - m - 1)! [(k - j)(1 + \zeta) + 2(p - \alpha)]}{2(p - \alpha)(p + k - j - m - 1)! p!} \alpha_{p+k-j} b_{p+k-j}, \]

the condition (3.1) can be written as

\[ 1 + \sum_{k=1}^{\infty} \left( \frac{b_k}{k!} + \sum_{j=0}^{k-1} \frac{\beta_j}{j!} A_{p+k-j} \right) z^k \neq 0 \]

which is equivalent to

\[ \left( 1 + \sum_{k=1}^{\infty} A_{p+k} z^k \right) \left( 1 + \sum_{k=1}^{\infty} \frac{\beta_k}{k!} z^k \right) \neq 0 \]

Since

\[ 1 + \sum_{k=1}^{\infty} \frac{\beta_k}{k!} z^k = e^{\beta z} \neq 0, \beta \in \mathbb{R}. \]

Thus (2.2) implies

\[ \left( 1 + \sum_{k=1}^{\infty} A_{p+k} z^k \right) \neq 0 \]

which by Theorem 1 proves that \( f(z) \in \mathcal{K}(p, g, \alpha, m). \)

**Theorem 4** Let \( g \in A_p \) with positive coefficients, \( m < p, 0 \leq \alpha < p, p > m, m \in \mathbb{N}_0 \), then \( TK(p, g, \alpha, m + 1) \subset TK(p, g, \alpha, m) \).

**Proof** Let \( f \in T_p \) of the form (1.2) belongs to \( TK(p, g, \alpha, m + 1) \) and \( g \in A_p \) of the form (1.3), then by Theorem 2, we get

\[ \sum_{k=1}^{\infty} \frac{(k + p)! (p + k - \alpha)}{(p - \alpha) p!} \varphi_k^{m+1} b_{p+k} \left| a_{p+k} \right| \leq 1, \]

Where

\[ \varphi_k^{m+1} = \frac{(p-m-2)!}{(p+k-m-2)!}. \]

By simple calculation, we see that \( \varphi_k^m \leq \varphi_k^{m+1} \) and hence,

\[ \sum_{k=1}^{\infty} \frac{(k + p)! (p + k - \alpha)}{(p - \alpha) p!} \varphi_k^m b_{p+k} \left| a_{p+k} \right| \leq \sum_{k=1}^{\infty} \frac{(k + p)! (p + k - \alpha)}{(p - \alpha) p!} \varphi_k^{m+1} b_{p+k} \left| a_{p+k} \right| \leq 1, \]

Which again by Theorem 2 it is proved that \( f(z) \in TK(p, g, \alpha, m) \). This proves Theorem 4.

4. **PARTIAL SUMS FOR THE CLASS \( K(p, g, \alpha, m) \).**

Following the earlier works by Silverman\(^8\), Silvia\(^7\), T. Rosy\(^4\), Zhi-Gang Wang\(^10\) and P. Srivastava\(^3\) on partial sums of analytic functions. We consider in this section partial sums of functions in the class \( K(p, g, \alpha, m) \) and \( K_\zeta(p, g, m) \) and obtain sharp lower bounds for the ratios of real part of \( f(z) \) to \( f_n(z) \) and \( f'(z) \) to \( f'_n(z) \).

Consider the partial sums \( f(z) \in A_p \) of of the form (1.1) be as follows:
\( f(z) = z^n \) and \( f_n(z) = z^n + \sum_{k=1}^{n-1} a_{p+k} z^{p+k}, n \geq 2. \) \( (4.1) \)

**Theorem 5** Let \( g(z) \in A_p \) be of the form (1.2) and for \( 0 \leq \alpha < p, m \in \mathbb{N}_0, \) the function \( f(z) \in \mathcal{A}_p \) of the form (1.1) satisfies

\[
\sum_{k=1}^{n-1} c_{p+k} |a_{p+k}| \leq 1, k \in \mathbb{N}_0
\]

Where \( c_{p+k} = \frac{(p-m-1)(p+k)(k+p-a)}{(p-a)(p+k-m-1)} b_{p+k}, \)

Then \( f \in K(p, g, \alpha, m) \) and

\[
\text{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{c_{p+n}} \quad (4.3)
\]

\[
\text{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{c_{p+n}}{1 + c_{p+n}} \quad (4.4)
\]

**Proof:** Let (4.1) holds, from Theorem 2, we get that \( f \in K(p, g, \alpha, m). \) Again it is easy to verify that \( c_{p+k} > c_{p+k} > 1, k \in \mathbb{N}_0, \)

From (4.2), we have

\[
\sum_{k=1}^{n-1} |a_{p+k}| + c_{p+n} \sum_{k=n}^{\infty} |a_{p+k}| \leq \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \leq 1. \quad (4.5)
\]

Set

\[
w_1 = c_{p+n} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{c_{p+n}} \right) \right\}
= 1 + \frac{c_{p+n} \sum_{k=n}^{\infty} a_{p+k} z^k}{1 + \sum_{k=1}^{n-1} a_{p+k} z^k}
\]

On using (4.5), we get

\[
\left| \frac{w_1(z) - 1}{w_1(z) + 1} \right| = \left| \frac{c_{p+n} \sum_{k=n}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^{n-1} a_{p+k} z^k + c_{p+n} \sum_{k=n}^{\infty} a_{p+k} z^k} \right|
\leq \left| \frac{c_{p+n} \sum_{k=n}^{\infty} |a_{p+k}|}{2 - 2 \sum_{k=1}^{n-1} |a_{p+k}| - c_{p+k} \sum_{k=n}^{\infty} |a_{p+k}|} \right| \leq 1,
\]

This readily yields the assertion (4.3) of Theorem 5.

Similarly, if we take

\[
w_2(z) = (1 + c_{p+n}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{p+n}}{1 + c_{p+n}} \right\}
= 1 - \frac{(c_{p+k} + 1) \sum_{k=1}^{\infty} a_{p+k} z^k}{1 + \sum_{k=1}^{\infty} a_{p+k} z^k}
\]

We get with the use of (4.5)

\[
\left| \frac{w_2(z) - 1}{w_2(z) + 1} \right| = \left| \frac{- (c_{p+n} + 1) \sum_{k=1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^{\infty} a_{p+k} z^k - (c_{p+k} + 1) \sum_{k=1}^{\infty} a_{p+k} z^k} \right|
\]

\[
\leq \frac{(c_{p+n} - 1) \sum_{k=n}^{\infty} |a_{p+k}|}{2 - 2 \sum_{k=1}^{n-1} |a_{p+k}| - (c_{p+n} - 1) \sum_{k=n}^{\infty} |a_{p+k}|} \leq 1,
\]

Which proves (4.4) of Theorem 5.

**Theorem 6** If \( f(z) \) of the form (1.1) satisfies the condition (4.1) then

\[
\text{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} > 1 - \frac{n+p}{c_{p+n}} ,
\]

(4.6)

\[
\text{Re}\left\{\frac{f_n(z)}{f'(z)}\right\} > \frac{c_{p+n}}{n+p+c_{p+n}} ,
\]

(4.7)

**Proof:** By setting

\[
w_3 = \frac{c_{p+n}}{p+n} \left(\frac{f'(z)}{f_n'(z)} - \left(1 - \frac{p+n}{c_{p+n}}\right)\right),
\]

\[
= 1 + \frac{\sum_{k=n}^{\infty} \frac{(p+k)}{p} a_{p+k} z^k}{1 + \sum_{k=1}^{n-1} \frac{(p+k)}{p} a_{p+k} z^k},
\]

We get

\[
\left|\frac{w_3(z) - 1}{w_3(z) + 1}\right| \leq \frac{\sum_{k=n}^{\infty} \frac{a_{p+k} z^k}{p+n} |a_{p+k}|}{2 - 2 \sum_{k=1}^{n-1} \frac{a_{p+k} z^k}{p+n} |a_{p+k}| - \frac{c_{p+k}}{p+n} \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}|},
\]

(4.8)

If

\[
\frac{c_{p+n}}{p+n} \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}| \leq 2 - 2 \sum_{k=1}^{n-1} \frac{p+k}{p} |a_{p+k}| - \frac{c_{p+k}}{p+n} \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}|
\]

Or,

\[
\sum_{k=1}^{n-1} \frac{1}{p} |a_{p+k}| + \frac{c_{p+n}}{p+n} \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}| \leq 1
\]

(4.9)

Since, the left hand side of (4.6) is bounded above by \( \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}| \) if

\[
\sum_{k=1}^{n-1} \frac{p+k}{p} |a_{p+k}| + \frac{c_{p+n}}{p+n} \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}| \leq \sum_{k=1}^{\infty} c_{p+k} |a_{p+k}|
\]

or,

\[
\sum_{k=1}^{n-1} \left(c_{p+k} - \frac{p+k}{p}\right) |a_{p+k}| + \sum_{k=n}^{\infty} \left(c_{p+k} - \frac{c_{p+n} (p+k)}{p+n}\right) |a_{p+k}| \geq 0
\]

This is true. This proves assertion (4.6) of Theorem 6.
Again by setting, \( w_4(z) = \left[ (n + p) + c_{n-p} \right] \left( \frac{f_n'(z)}{f(z)} - \frac{c_{n+p}}{n + p + c_{n+p}} \right) \)

\[
1 - \frac{1 + \frac{c_{n+p}}{n + p} \sum_{k=1}^{\infty} \frac{p + k}{p} a_{p+k} z^k}{1 + \sum_{k=1}^{\infty} \frac{p + k}{p} a_{p+k} z^k}
\]

Now,

\[
\frac{|w_4(z) - 1|}{|w_4(z) + 1|} \leq \frac{\left( 1 + \frac{c_{n+p}}{n + p} \right) \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}|}{2 - 2 \sum_{k=1}^{n-1} \frac{p+k}{p} |a_{p+k}| - \left( 1 + \frac{c_{n+p}}{n + p} \right) \sum_{k=n}^{\infty} \frac{p+k}{p} |a_{p+k}|} \leq 1
\]

which leads to the result (4.7).

**REFERENCES**